211en02

2

EXPONENTS AND RADICALS

We have learnt about multiplication of two or more real numbers in the earlier lesson. You can very easily write the following

$$4 \times 4 \times 4 = 64, 11 \times 11 \times 11 \times 11 = 14641 \text{ and}$$

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 256$$

Think of the situation when 13 is to be multiplied 15 times. How difficult is it to write?

$$13 \times 13 \times 13 \times \text{................}15 \text{ times?}$$

This difficulty can be overcome by the introduction of exponential notation. In this lesson, we shall explain the meaning of this notation, state and prove the laws of exponents and learn to apply these. We shall also learn to express real numbers as product of powers of prime numbers.

In the next part of this lesson, we shall give a meaning to the number $a^{1/q}$ as qth root of *a*. We shall introduce you to radicals, index, radicand etc. Again, we shall learn the laws of radicals and find the simplest form of a radical. We shall learn the meaning of the term “rationalising factor’ and rationalise the denominators of given radicals.

OBJECTIVES

After studying this lesson, you will be able to

- *write a repeated multiplication in exponential notation and vice-versa;*
- *identify the base and exponent of a number written in exponential notation;*
- *express a natural number as a product of powers of prime numbers uniquely;*
- *state the laws of exponents;*
- *explain the meaning of* a^0, a^{-m} *and* $a^{\frac{p}{q}}$;
- *simplify expressions involving exponents, using laws of exponents;*

- *identify radicals from a given set of irrational numbers;*
- *identify index and radicand of a surd;*
- *state the laws of radicals (or surds);*
- *express a given surd in simplest form;*
- *classify similar and non-similar surds;*
- *reduce surds of different orders to those of the same order;*
- *perform the four fundamental operations on surds;*
- *arrange the given surds in ascending/descending order of magnitude;*
- *find a rationalising factor of a given surd;*
- *rationalise the denominator of a given surd of the form* $\frac{1}{a+b\sqrt{x}}$ and $\frac{1}{\sqrt{x}+\sqrt{y}}$,

  where x and y are natural numbers and a and b are integers;
- *simplify expressions involving surds.*

EXPECTED BACKGROUND KNOWLEDGE

- Prime numbers
- Four fundamental operations on numbers
- Rational numbers
- Order relation in numbers.

2.1 EXPONENTIAL NOTATION

Consider the following products:

(i) 7×7 (ii) $3 \times 3 \times 3$ (iii) $6 \times 6 \times 6 \times 6 \times 6$

In (i), 7 is multiplied twice and hence 7×7 is written as 7^2.

In (ii), 3 is multiplied three times and so $3 \times 3 \times 3$ is written as 3^3.

In (iii), 6 is multiplied five times, so $6 \times 6 \times 6 \times 6 \times 6$ is written as 6^{5}.

7^2 is read as "7 raised to the power 2" or "second power of 7". Here, 7 is called base and 2 is called exponent (or index)

Similarly, 3^3 is read as "3 raised to the power 3"or "third power of 3". Here, 3 is called the base and 3 is called exponent.

Similarly, 6^5 is read as "6 raised to the power 5"or "Fifth power of 6". Again 6 is base and 5 is the exponent (or index).

From the above, we say that

The notation for writing the product of a number by itself several times is called the Exponential Notation or Exponential Form.

Thus, $5 \times 5 \times \ldots.$ 20 times $= 5^{20}$ and $(-7) \times (-7) \times \ldots.$ 10 times $= (-7)^{10}$

In 5^{20}, 5 is the base and exponent is 20.

In $(-7)^{10}$, base is -7 and exponent is 10.

Similarly, exponential notation can be used to write precisely the product of a ratioinal number by itself a number of times.

Thus, $$\frac{3}{5} \times \frac{3}{5} \times \ldots\ldots\text{16 times} = \left(\frac{3}{5}\right)^{16}$$

and $$\left(-\frac{1}{3}\right) \times \left(-\frac{1}{3}\right) \times \ldots\ldots\text{10 times} = \left(-\frac{1}{3}\right)^{10}$$

In general, if a is a rational number, multiplied by itself m times, it is written as a^m.

Here again, a is called the base and m is called the exponent

Let us take some examples to illustrate the above discussion:

Example 2.1: Evaluate each of the following:

(i) $\left(\frac{2}{7}\right)^3$ (ii) $\left(-\frac{3}{5}\right)^4$

Solution: (i) $$\left(\frac{2}{7}\right)^3 = \frac{2}{7} \times \frac{2}{7} \times \frac{2}{7} = \frac{(2)^3}{(7)^3} = \frac{8}{343}$$

(ii) $$\left(-\frac{3}{5}\right)^4 = \left(-\frac{3}{5}\right)\left(-\frac{3}{5}\right)\left(-\frac{3}{5}\right)\left(-\frac{3}{5}\right) = \frac{(-3)^4}{(5)^5} = \frac{81}{625}$$

Example 2.2: Write the following in exponential form:

(i) $(-5) \times (-5) \times (-5) \times (-5) \times (-5) \times (-5) \times (-5)$

(ii) $\left(\frac{3}{11}\right) \times \left(\frac{3}{11}\right) \times \left(\frac{3}{11}\right) \times \left(\frac{3}{11}\right)$

Solution: (i) $(-5) \times (-5) \times (-5) \times (-5) \times (-5) \times (-5) \times (-5) = (-5)^7$

(ii) $$\left(\frac{3}{11}\right) \times \left(\frac{3}{11}\right) \times \left(\frac{3}{11}\right) \times \left(\frac{3}{11}\right) = \left(\frac{3}{11}\right)^4$$

Example 2.3: Express each of the following in exponential notation and write the base and exponent in each case.

(i) 4096 (ii) $\frac{125}{729}$ (iii) -512

Solution: (i) $4096 = 4 \times 4 \times 4 \times 4 \times 4 \times 4$

$= (4)^6$

Alternatively $4096 = (2)^{12}$

Base = 2, exponent = 12

Here, base = 4 and exponent = 6

(ii) $\frac{125}{729} = \frac{5}{9} \times \frac{5}{9} \times \frac{5}{9} = \left(\frac{5}{9}\right)^3$

Here, base = $\left(\frac{5}{9}\right)$ and exponent = 3

(iii) $512 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^9$

Here, base = 2 and exponent = 9

Example 2.4: Simplify the following:

$$\left(\frac{3}{2}\right)^3 \times \left(\frac{4}{3}\right)^4$$

Solution:

$$\left(\frac{3}{2}\right)^3 = \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} = \frac{3^3}{2^3}$$

Similarly

$$\left(\frac{4}{3}\right)^4 = \frac{4^4}{3^4}$$

$$\left(\frac{3}{2}\right)^3 \times \left(\frac{4}{3}\right)^4 = \frac{3^3}{2^3} \times \frac{4^4}{3^4}$$

$$= \frac{3^3}{8} \times \frac{16 \times 16}{3^4} = \frac{32}{3}$$

Example 2.5: Write the reciprocal of each of the following and express them in exponential form:

(i) 3^5 (ii) $\left(\frac{3}{4}\right)^2$ (iii) $\left(-\frac{5}{6}\right)^9$

Solution: (i) $3^5 = 3 \times 3 \times 3 \times 3 \times 3$

$= 243$

$$\therefore \text{ Reciprocal of } 3^5 = \frac{1}{243} = \left(\frac{1}{3}\right)^5$$

(ii) $$\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2}$$

$$\therefore \text{ Reciprocal of } \left(\frac{3}{4}\right)^2 = \frac{4^2}{3^2} = \left(\frac{4}{3}\right)^2$$

(iii) $$\left(-\frac{5}{6}\right)^9 = \frac{(-5)^9}{6^9}$$

$$\therefore \text{ Reciprocal of } \left(-\frac{5}{6}\right)^9 = \frac{-6^9}{5^9} = \left(\frac{-6}{5}\right)^9$$

From the above example, we can say that if $\frac{p}{q}$ is any non-zero rational number and m is any positive integer, then the reciprocal of $\left(\frac{p}{q}\right)^m$ is $\left(\frac{q}{p}\right)^m$.

CHECK YOUR PROGRESS 2.1

1. Write the following in exponential form:

   (i) $(-7) \times (-7) \times (-7) \times (-7)$

   (ii) $\left(\frac{3}{4}\right) \times \left(\frac{3}{4}\right) \times \ldots$ 10 times

   (iii) $\left(-\frac{5}{7}\right) \times \left(-\frac{5}{7}\right) \times \ldots$ 20 times

2. Write the base and exponent in each of the following:

(i) $(-3)^5$ (ii) $(7)^4$ (iii) $\left(-\dfrac{2}{11}\right)^8$

3. Evaluate each of the following

(i) $\left(\dfrac{3}{7}\right)^4$ (ii) $\left(\dfrac{-2}{9}\right)^4$ (iii) $\left(-\dfrac{3}{4}\right)^3$

4. Simplify the following:

(i) $\left(\dfrac{7}{3}\right)^5 \times \left(\dfrac{3}{7}\right)^6$

(ii) $\left(-\dfrac{5}{6}\right)^2 \div \left(-\dfrac{3}{5}\right)^2$

5. Find the reciprocal of each of the following:

(i) 3^5 (ii) $(-7)^4$ (iii) $\left(-\dfrac{3}{5}\right)^4$

2.2 PRIME FACTORISATION

Recall that any composite number can be expressed as a product of prime numbers. Let us take the composite numbers 72, 760 and 7623.

(i) $72 = 2 \times 2 \times 2 \times 3 \times 3$

$= 2^3 \times 3^2$

(ii) $760 = 2 \times 2 \times 2 \times 5 \times 19$

$= 2^3 \times 5^1 \times 19^1$

(iii) $7623 = 3 \times 3 \times 7 \times 11 \times 11$

$= 3^2 \times 7^1 \times 11^2$

$$\begin{array}{r|l} 2 & 72 \\ \hline 2 & 36 \\ \hline 2 & 18 \\ \hline 3 & 9 \\ \hline & 3 \end{array} \qquad \begin{array}{r|l} 3 & 7623 \\ \hline 3 & 2541 \\ \hline 7 & 847 \\ \hline 11 & 121 \\ \hline & 11 \end{array} \qquad \begin{array}{r|l} 2 & 760 \\ \hline 2 & 380 \\ \hline 2 & 190 \\ \hline 5 & 95 \\ \hline & 19 \end{array}$$

We can see that any natural number, other than 1, can be expressed as a product of powers of prime numbers in a unique manner, apart from the order of occurrence of factors. Let us consider some examples

Example 2.6: Express 24300 in exponential form.

Solution: $24300 = 3 \times 3 \times 3 \times 3 \times 2 \times 2 \times 5 \times 5 \times 3$

$\therefore$ 24300 = $2^2 \times 3^5 \times 5^2$

Example 2.7: Express 98784 in exponential form.

Solution:

2	98784
2	49392
2	24696
2	12348
2	6174
3	3087
3	1029
7	343
7	49
	7

$\therefore$ 98784 = $2^5 \times 3^2 \times 7^3$

CHECK YOUR PROGRESS 2.2

1. Express each of the following as a product of powers of primes, i.e, in exponential form:

   (i) 429 (ii) 648 (iii) 1512

2. Express each of the following in exponential form:

   (i) 729 (ii) 512 (iii) 2592

   (iv) $\frac{1331}{4096}$ (v) $-\frac{243}{32}$

2.3 LAWS OF EXPONENTS

Consider the following

(i) $3^2 \times 3^3 = (3 \times 3) \times (3 \times 3 \times 3) = (3 \times 3 \times 3 \times 3 \times 3)$

$= 3^5 = 3^{2+3}$

(ii) $(-7)^2 \times (-7)^4 = [(-7) \times (-7)] \times [(-7) \times (-7) \times (-7) \times (-7)]$

$= [(-7) \times (-7) \times (-7) \times (-7) \times (-7) \times (-7)]$

$= (-7)^6 = (-7)^{2+4}$

(iii) $\left(\frac{3}{4}\right)^3 \times \left(\frac{3}{4}\right)^4 = \left(\frac{3}{4} \times \frac{3}{4} \times \frac{3}{4}\right) \times \left(\frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4}\right)$

$$=\left(\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}\right)$$

$$=\left(\frac{3}{4}\right)^7=\left(\frac{3}{4}\right)^{3+4}$$

(iv) $a^3 \times a^4 = (a \times a \times a) \times (a \times a \times a \times a) = a^7 = a^{3+4}$

From the above examples, we observe that

Law 1: If a is any non-zero rational number and m and n are two positive integers, then

$$a^m \times a^n = a^{m+n}$$

Example 2.8: Evaluate $\left(-\frac{3}{2}\right)^3 \times \left(-\frac{3}{2}\right)^5$.

Solution: Here $a = -\frac{3}{2}$, $m = 3$ and $n = 5$.

$$\therefore \left(-\frac{3}{2}\right)^3 \times \left(-\frac{3}{2}\right)^5 = \left(-\frac{3}{2}\right)^{3+5} = \left(-\frac{3}{2}\right)^8 = \frac{6561}{256}$$

Example 2.9: Find the value of

$$\left(\frac{7}{4}\right)^2 \times \left(\frac{7}{4}\right)^3$$

Solution: As before,

$$\left(\frac{7}{4}\right)^2 \times \left(\frac{7}{4}\right)^3 = \left(\frac{7}{4}\right)^{2+3} = \left(\frac{7}{4}\right)^5 = \frac{16807}{1024}$$

Now study the following:

(i) $7^5 \div 7^3 = \frac{7^5}{7^3} = \frac{7\times7\times7\times7\times7}{7\times7\times7} = 7\times7 = 7^2 = 7^{5-3}$

(ii) $(-3)^7 \div (-3)^4 = \frac{(-3)^7}{(-3)^4} = \frac{(-3)\times(-3)\times(-3)\times(-3)\times(-3)\times(-3)\times(-3)}{(-3)\times(-3)\times(-3)\times(-3)}$

$$= (-3)(-3)(-3) = (-3)^3 = (-3)^{7-4}$$

From the above, we can see that

Law 2: If a is any non-zero rational number and m and n are positive integers ($m > n$), then

$$a^m \div a^n = a^{m-n}$$

Example 2.10: Find the value of $\left(\frac{35}{25}\right)^{16} \div \left(\frac{35}{25}\right)^{13}$.

Solution: $\left(\frac{35}{25}\right)^{16} \div \left(\frac{35}{25}\right)^{13}$

$$= \left(\frac{35}{25}\right)^{16-13} = \left(\frac{35}{25}\right)^{3} = \left(\frac{7}{5}\right)^{3} = \frac{343}{125}$$

In Law 2, $m < n \Rightarrow n > m$,

then $$a^m \div a^n = a^{-(n-m)} = \frac{1}{a^{m-n}}$$

Law 3: When $n > m$

$$a^m \div a^n = \frac{1}{a^{m-n}}$$

Example 2.11: Find the value of $\left(\frac{3}{7}\right)^{6} \div \left(\frac{3}{7}\right)^{9}$

Solution: Here $a = \frac{3}{7}$, $m = 6$ and $n = 9$.

$$\therefore \left(\frac{3}{7}\right)^{6} \div \left(\frac{3}{7}\right)^{9} = \left(\frac{3}{7}\right)^{\frac{1}{9-6}}$$

$$= \frac{7^3}{3^3} = \frac{343}{27}$$

Let us consider the following:

(i) $\left(3^3\right)^2 = 3^3 \times 3^3 = 3^{3+3} = 3^6 = 3^{3\times 2}$

(ii) $\left[\left(\frac{3}{7}\right)^2\right]^5 = \left(\frac{3}{7}\right)^2 \times \left(\frac{3}{7}\right)^2 \times \left(\frac{3}{7}\right)^2 \times \left(\frac{3}{7}\right)^2 \times \left(\frac{3}{7}\right)^2$

$$\left(\frac{3}{7}\right)^{2+2+2+2+2} = \left(\frac{3}{7}\right)^{10} = \left(\frac{3}{7}\right)^{2\times 5}$$

From the above two cases, we can infer the following:

Law 4: If a is any non-zero rational number and m and n are two positive integers, then

$$\left(a^m\right)^n = a^{mn}$$

Let us consider an example.

Example 2.12: Find the value of $\left[\left(\frac{2}{5}\right)^2\right]^3$

Solution: $\left[\left(\frac{2}{5}\right)^2\right]^3 = \left[\frac{2}{5}\right]^{2\times 3} = \left(\frac{2}{5}\right)^6 = \frac{64}{15625}$

2.3.1 Zero Exponent

Recall that $a^m \div a^n = a^{m-n}$, if $m > n$

$$= \frac{1}{a^{n-m}}, \text{ if } n > m$$

Let us consider the case, when $m = n$

$$\therefore\ a^m \div a^m = a^{m-m}$$

$$\Rightarrow \frac{a^m}{a^m} = a^0$$

$$\Rightarrow 1 = a^0$$

Thus, we have another important law of exponents,.

Law 5: If a is any rational number other than zero, then $a^o = 1$.

Example 2.13: Find the value of

(i) $\left(\frac{2}{7}\right)^0$ (ii) $\left(\frac{-3}{4}\right)^0$

Solution: (i) Using $a^0 = 1$, we get $\left(\frac{2}{7}\right)^0 = 1$

(ii) Again using $a^0 = 1$, we get $\left(\frac{-3}{4}\right)^0 = 1.$

CHECK YOUR PROGRESS 2.3

1. Simplify and express the result in exponential form:

   (i) $(7)^2 \times (7)^3$ (ii) $\left(\frac{3}{4}\right)^3 \times \left(\frac{3}{4}\right)^2$ (iii) $\left(-\frac{7}{8}\right)^1 \times \left(-\frac{7}{8}\right)^2 \times \left(-\frac{7}{8}\right)^3$

2. Simplify and express the result in exponential form:

   (i) $(-7)^9 \div (-7)^7$ (ii) $\left(\frac{3}{4}\right)^8 \div \left(\frac{3}{4}\right)^2$ (iii) $\left(\frac{-7}{3}\right)^{18} \div \left(\frac{-7}{3}\right)^3$

3. Simplify and express the result in exponential form:

   (i) $\left(2^6\right)^3$ (ii) $\left[\left(\frac{3}{4}\right)^3\right]^2$ (iii) $\left[\left(-\frac{5}{9}\right)^3\right]^5$

   (iv) $\left(\frac{11}{3}\right)^5 \times \left(\frac{15}{7}\right)^0$ (v) $\left(-\frac{7}{11}\right)^0 \times \left(-\frac{7}{11}\right)^3$

4. Which of the following statements are true?

   (i) $7^3 \times 7^3 = 7^6$ (ii) $\left(\frac{3}{11}\right)^5 \times \left(\frac{3}{11}\right)^2 = \left(\frac{3}{11}\right)^7$

   (iii) $\left[\left(\frac{4}{9}\right)^5\right]^4 = \left(\frac{4}{9}\right)^9$ (iv) $\left[\left(\frac{3}{19}\right)^6\right]^2 = \left(\frac{3}{19}\right)^8$

   (v) $\left(\frac{3}{11}\right)^0 = 0$ (vi) $\left(-\frac{3}{2}\right)^2 = -\frac{9}{4}$

   (vii) $\left(\frac{8}{15}\right)^5 \times \left(\frac{7}{6}\right)^0 = \left(\frac{8}{15}\right)^5$

2.4 NEGATIVE INTEGERS AS EXPONENTS

i) We know that the reciprocal of 5 is $\frac{1}{5}$. We write it as 5^{-1} and read it as 5 raised to power –1.

ii) The reciprocal of (–7) is $-\frac{1}{7}$. We write it as $(-7)^{-1}$ and read it as (–7) raised to the power –1.

iii) The reciprocal of $5^2 = \frac{1}{5^2}$. We write it as 5^{-2} and read it as '5 raised to the power (–2)'.

From the above all, we get

If a is any non-zero rational number and m is any positive integer, then the reciprocal of a^m $\left(i.e. \frac{1}{a^m}\right)$ is written as a^{-m} and is read as 'a raised to the power (–m)'. Therefore,

$$\boxed{\frac{1}{a^m} = a^{-m}}$$

Let us consider an example.

Example 2.14: Rewrite each of the following with a positive exponent:

$$\text{(i)} \left(\frac{3}{8}\right)^{-2} \qquad \text{(ii)} \left(-\frac{4}{7}\right)^{-7}$$

Solution:

$$\text{(i)} \left(\frac{3}{8}\right)^{-2} = \frac{1}{\left(\frac{3}{8}\right)^2} = \frac{1}{\frac{3^2}{8^2}} = \frac{8^2}{3^2} = \left(\frac{8}{3}\right)^2$$

$$\text{(ii)} \left(-\frac{4}{7}\right)^{-7} = \frac{1}{\left(-\frac{4}{7}\right)^7} = \frac{7^7}{(-4)^7} = \left(-\frac{7}{4}\right)^7$$

From the above example, we get the following result:

If $\frac{p}{q}$ is any non-zero rational number and m is any positive integer, then

$$\left(\frac{p}{q}\right)^{-m} = \frac{q^m}{p^m} = \left(\frac{q}{p}\right)^m.$$

2.5 LAWS OF EXPONENTS FOR INTEGRAL EXPONENTS

After giving a meaning to negative integers as exponents of non-zero rational numbers, we can see that laws of exponents hold good for negative exponents also.

For example.

(i) $$\left(\frac{3}{5}\right)^{-4}\times\left(\frac{3}{5}\right)^{3}=\frac{1}{\left(\frac{3}{5}\right)^{4}}\times\left(\frac{3}{5}\right)^{3}=\frac{3^{3-4}}{5}$$

(ii) $$\left(-\frac{2}{3}\right)^{-2}\times\left(-\frac{2}{3}\right)^{-3}=\frac{1}{\left(-\frac{2}{3}\right)^{2}}\times\frac{1}{\left(-\frac{2}{3}\right)^{3}}=\frac{1}{\left(-\frac{2}{3}\right)^{2+3}}=\left(-\frac{2}{3}\right)^{-2-3}$$

(iii) $$\left(-\frac{3}{4}\right)^{-3}\div\left(-\frac{3}{4}\right)^{-7}=\frac{1}{\left(-\frac{3}{4}\right)^{3}}\div\frac{1}{\left(-\frac{3}{4}\right)^{7}}=\frac{1}{\left(-\frac{3}{4}\right)^{3}}\times\left(-\frac{3}{4}\right)^{7}=\left(-\frac{3}{4}\right)^{7-3}$$

(iv) $$\left(\left(\frac{2}{7}\right)^{-2}\right)^{3}=\left[\left(\frac{7}{2}\right)^{2}\right]^{3}=\left(\frac{7}{2}\right)^{6}=\left(\frac{2}{7}\right)^{-6}=\left(\frac{2}{7}\right)^{-2\times 3}$$

Thus, from the above results, we find that laws 1 to 5 hold good for negative exponents also.

$\therefore$ For any non-zero rational numbers a and b and any integers m and n,

1. $a^m \times a^n \quad = a^{m+n}$
2. $a^m \div a^n \quad = a^{m-n}$ if $m > n$

   $= a^{n-m}$ if $n > m$
3. $(a^m)^n = a^{mn}$
4. $(a \times b)^m = a^m \times b^m$

CHECK YOUR PROGRESS 2.4

1. Express $\left(\frac{-3}{7}\right)^{-2}$ as a rational number of the form $\frac{p}{q}$:

2. Express as a power of rational number with positive exponent:

(i) $\left(\frac{3}{7}\right)^{-4}$ (ii) $12^5 \times 12^{-3}$ (iii) $\left[\left(\frac{3}{13}\right)^{-3}\right]^4$

3. Express as a power of a rational number with negative index:

(i) $\left(\frac{3}{7}\right)^{4}$ (ii) $\left[(7)^2\right]^5$ (iii) $\left[\left(-\frac{3}{4}\right)^{2}\right]^5$

4. Simplify:

(i) $\left(\frac{3}{2}\right)^{-3} \times \left(\frac{3}{2}\right)^{7}$ (ii) $\left(-\frac{2}{3}\right)^{-3} \times \left(-\frac{2}{3}\right)^{4}$ (iii) $\left(-\frac{7}{5}\right)^{-4} \div \left(-\frac{7}{5}\right)^{-7}$

5. Which of the following statements are true?

(i) $a^{-m} \times a^n = a^{-m-n}$

(ii) $(a^{-m})^n = a^{-mn}$

(iii) $a^m \times b^m = (ab)^m$

(iv) $a^m \div b^m = \left(\frac{a}{b}\right)^m$

(v) $a^{-m} \times a^o = a^m$

2.6 MEANING OF $a^{p/q}$

You have seen that for all integral values of m and n,

$$a^m \times a^n = a^{m+n}$$

What is the method of defining $a^{1/q}$, if a is positive rational number and q is a natural number.

Consider the multiplication

$$\underbrace{a^{\frac{1}{q}} \times a^{\frac{1}{q}} \times a^{\frac{1}{q}} \ldots\ldots\ldots \times a^{\frac{1}{q}}}_{q \text{ times}} = a^{\frac{1}{q}+\frac{1}{q}+\frac{1}{q}+\ldots\ldots q \text{ times}}$$

$$= a^{\frac{q}{q}} = a$$

In other words, the qth power of $a^{\frac{1}{q}} = a$ or

in other words $a^{\frac{1}{q}}$ is the qth root of a and is written as $\sqrt[q]{a}$.

For example,

$$7^{\frac{1}{4}} \times 7^{\frac{1}{4}} \times 7^{\frac{1}{4}} \times 7^{\frac{1}{4}} = 7^{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = 7^{\frac{4}{4}} = 7^1 = 7$$

or $7^{\frac{1}{4}}$ is the fourth root of 7 and is written as $\sqrt[4]{7}$,

Let us now define rational powers of a

If a is a positive real number, p is an integer and q is a natural number, then

$$a^{\frac{p}{q}} = \sqrt[q]{a^p}$$

We can see that

$$\underbrace{a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \ldots\ldots \times a^{\frac{p}{q}}}_{q \text{ times}} = a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}+\ldots\ldots q \text{ times}} = a^{\frac{p}{q}.q} = a^p$$

$$\therefore a^{\frac{p}{q}} = \sqrt[q]{a^p}$$

$\therefore$ $a^{p/q}$ is the qth root of a^p

Consequently, $7^{\frac{2}{3}}$ is the cube root of 7^2.

Let us now write the laws of exponents for rational exponents:

(i) $a^m \times a^n = a^{m+n}$

(ii) $a^m \div a^n = a^{m-n}$

(iii) $(a^m)^n = a^{mn}$

(iv) $(ab)^m = a^m b^m$

(v) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Let us consider some examples to verify the above laws:

Example 2.15: Find the value of

(i) $(625)^{\frac{1}{4}}$ (ii) $(243)^{\frac{2}{5}}$ (iii) $\left(\frac{16}{81}\right)^{-3/4}$

Solution:

(i) $(625)^{\frac{1}{4}} = (5\times5\times5\times5)^{\frac{1}{4}} = \left(5^4\right)^{\frac{1}{4}} = 5^{4\times\frac{1}{4}} = 5$

(ii) $(243)^{\frac{2}{5}} = (3\times3\times3\times3\times3)^{\frac{2}{5}} = \left(3^5\right)^{\frac{2}{5}} = 3^{5\times\frac{2}{5}} = 3^2 = 9$

(iii) $\left(\dfrac{16}{81}\right)^{\frac{-3}{4}} = \left(\dfrac{2\times2\times2\times2}{3\times3\times3\times3}\right)^{\frac{-3}{4}}$

$$= \left[\left(\frac{2}{3}\right)^4\right]^{\frac{-3}{4}} = \left(\frac{2}{3}\right)^{4\times\left(\frac{-3}{4}\right)} = \left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$$

CHECK YOUR PROGRESS 2.5

1. Simplify each of the following:

   (i) $(16)^{3/4}$ (ii) $\left(\dfrac{27}{125}\right)^{-\frac{2}{3}}$

2. Simplify each of the following:

   (i) $(625)^{-\frac{1}{4}} \div (25)^{-\frac{1}{2}}$

   (ii) $\left(\dfrac{7}{8}\right)^{-\frac{1}{4}} \times \left(\dfrac{7}{8}\right)^{\frac{1}{2}} \times \left(\dfrac{7}{8}\right)^{\frac{3}{4}}$

   (iii) $\left(\dfrac{13}{16}\right)^{-\frac{3}{4}} \times \left(\dfrac{13}{16}\right)^{\frac{1}{4}} \times \left(\dfrac{13}{16}\right)^{\frac{3}{2}}$

2.7 SURDS

We have read in first lesson that numbers of the type $\sqrt{2}, \sqrt{3}$ and $\sqrt{5}$ are all irrational numbers. We shall now study irrational numbers of a particular type called **radicals** or **surds**.

A surd is defined as a positive irrational number of the type $\sqrt[n]{x}$, where it is not possible to find exactly the nth root of x, where x is a positive rational number.

The number $\sqrt[n]{x}$ is a surd if and only if

(i) it is an irrational number

(ii) it is a root of the positive rational number

2.7.1 Some Terminology

In the surd $\sqrt[n]{x}$, the symbol $\sqrt{\ }$ is called a **radical sign**. The index 'n' is called the **order** of the surd and *x* is called the **radicand**.

Note: i) When order of the surd is not mentioned, it is taken as 2. For example, order of $\sqrt{7}$ $\left(=\sqrt[2]{7}\right)$ is 2.

ii) $\sqrt[3]{8}$ is not a surd as its value can be determined as 2 which is a rational.

iii) $\sqrt{2+\sqrt{2}}$, although an irrational number, is not a surd because it is the square root of an irrational number.

2.8 PURE AND MIXED SURD

i) A surd, with rational factor is 1 only, other factor being rrational is called a **pure surd**.

For example, $\sqrt[5]{16}$ and $\sqrt[3]{50}$ are pure surds.

ii) A surd, having rational factor other than 1 alongwith the irrational factor, is called a mixed surd.

For example, $2\sqrt{3}$ and $3\sqrt[3]{7}$ are mixed surds.

2.9 ORDER OF A SURD

In the surd $5\sqrt[3]{4}$, 5 is called the **co-efficient** of the surd, 3 is the **order** of the surd and 4 is the **radicand**. Let us consider some examples:

Example 2.16: State which of the following are surds?

(i) $\sqrt{49}$ (ii) $\sqrt{96}$ (iii) $\sqrt[3]{81}$ (iv) $\sqrt[3]{256}$

Solution: (i) $\sqrt{49} = 7$, which is a rational number.

$\therefore \sqrt{49}$ is not a surd.

(ii) $\sqrt{96} = \sqrt{4 \times 4 \times 6} = 4\sqrt{6}$

$\therefore \sqrt{96}$ is an irrational number.

$\Rightarrow \sqrt{96}$ is a surd.

(iii) $\sqrt[3]{81} = \sqrt[3]{3 \times 3 \times 3 \times 3} = 3\sqrt[3]{3}$, which is irrational

$\therefore \sqrt[3]{81}$ is a surd.

(iv) $\sqrt[3]{256} = \sqrt[3]{4 \times 4 \times 4 \times 4} = 4\sqrt[3]{4}$

$\therefore \sqrt[3]{256}$ is irrational.

$\Rightarrow \sqrt[3]{256}$ is a surd

$\therefore$ (ii), (iii) and (iv) are surds.

Example 2.17: Find "index" and "radicand" in each of the following:

(i) $\sqrt[5]{117}$ (ii) $\sqrt{162}$ (iii) $\sqrt[4]{213}$ (iv) $\sqrt[4]{214}$

Solution: (i) index is 5 and radicand is 117.

(ii) index is 2 and radicand is 162.

(iii) index is 4 and radicand is 213.

(iv) index is 4 and radicand is 214.

Example 2.18: Identify "pure" and "mixed" surds from the following:

(i) $\sqrt{42}$ (ii) $4\sqrt[3]{18}$ (iii) $2\sqrt[4]{98}$

Solution: (i) $\sqrt{42}$ is a pure surd.

(ii) $4\sqrt[3]{18}$ is a mixed surd.

(iii) $2\sqrt[4]{98}$ is a mixed surd.

2.10 LAWS OF RADICALS

Given below are Laws of Radicals: (without proof):

(i) $\left[\sqrt[n]{a}\right]^n = a$

(ii) $\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$

(iii) $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$

where a and b are positive rational numbers and n is a positive integer.

Let us take some examples to illustrate.

Example 2.19: Which of the following are surds and which are not? Use laws of radicals to ascertain.

(i) $\sqrt{5}\times\sqrt{80}$ (ii) $2\sqrt{15}\div 4\sqrt{10}$

(iii) $\sqrt[3]{4}\times\sqrt[3]{16}$ (iv) $\sqrt{32}\div\sqrt{27}$

Solution: (i) $\sqrt{5}\times\sqrt{80} = \sqrt{5\times 80} = \sqrt{400} = 20\cdot$

which is a rational number.

$\therefore$ $\sqrt{5}\times\sqrt{80}$ is not a surd.

(ii) $2\sqrt{15}\div 4\sqrt{10} = \dfrac{2\sqrt{15}}{4\sqrt{10}} = \dfrac{\sqrt{15}}{2\sqrt{10}}$

$= \dfrac{\sqrt{15}}{\sqrt{2\times 2\times 10}} = \dfrac{\sqrt{15}}{\sqrt{40}} = \sqrt{\dfrac{3}{8}}$, which is irrational.

$\therefore$ $2\sqrt{15}\div 4\sqrt{10}$ is a surd.

(iii) $\sqrt[3]{4}\times\sqrt[3]{16} = \sqrt[3]{64} = 4 \Rightarrow$ It is not a surd.

(iv) $\sqrt{32}\div\sqrt{27} = \dfrac{\sqrt{32}}{\sqrt{27}} = \sqrt{\dfrac{32}{27}}$, which is irrational

$\therefore$ $\sqrt{32}\div\sqrt{27}$ is a surd.

CHECK YOUR PROGRESS 2.6

1. For each of the following, write index and the radicand:

(i) $\sqrt[4]{64}$ (ii) $\sqrt[6]{343}$ (iii) $\sqrt{119}$

2. State which of the following are surds:

(i) $\sqrt[3]{64}$ (ii) $\sqrt[4]{625}$ (iii) $\sqrt[6]{216}$

(iv) $\sqrt{5}\times\sqrt{45}$ (v) $3\sqrt{2}\times5\sqrt{6}$

3. Identify pure and mixed surds out of the following:

(i) $\sqrt{32}$ (ii) $2\sqrt[3]{12}$ (iii) $13\sqrt[3]{91}$ (iv) $\sqrt{35}$

2.11 LAWS OF SURDS

Recall that the surds can be expressed as numbers with fractional exponents. Therefore, laws of indices studied in this lesson before, are applicable to them also. Let us recall them here:

(i) $\sqrt[n]{x}\,.\,\sqrt[n]{y}=\sqrt[n]{xy}$ or $x^{\frac{1}{n}}.y^{\frac{1}{n}}=(xy)^{\frac{1}{n}}$

(ii) $\dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}=\sqrt[n]{\dfrac{x}{y}}$ or $\dfrac{x^{\frac{1}{n}}}{y^{\frac{1}{n}}}=\left(\dfrac{x}{y}\right)^{\frac{1}{n}}$

(iii) $\sqrt[m]{\sqrt[n]{x}}=\sqrt[mn]{x}=\sqrt[n]{\sqrt[m]{x}}$ or $\left(x^{\frac{1}{n}}\right)^{\frac{1}{m}}=x^{\frac{1}{mn}}=\left(x^{\frac{1}{m}}\right)^{\frac{1}{n}}$

(iv) $\sqrt[n]{x^m}=x^{\frac{m}{n}}$ or $\left(x^m\right)^{\frac{1}{n}}=x^{\frac{m}{n}}$

(v) $\sqrt[n]{x^p}=\sqrt[mn]{x^{pn}}$ or $\left(x^p\right)^{\frac{1}{m}}=x^{\frac{p}{m}}=x^{\frac{pn}{mn}}=\left(x^{pn}\right)^{\frac{1}{mn}}$

Here, x and y are positive rational numbers and m, n and p are positive integers.

Let us illustrate these laws by examples:

(i) $\sqrt[3]{3}\ \sqrt[3]{8}=3^{\frac{1}{3}}\times8^{\frac{1}{3}}=(24)^{\frac{1}{3}}=\sqrt[3]{24}=\sqrt[3]{3\times8}$

(ii) $\dfrac{(5)^{\frac{1}{3}}}{(9)^{\frac{1}{3}}}=\left(\dfrac{5}{9}\right)^{\frac{1}{3}}=\sqrt[3]{\dfrac{5}{9}}$

(iii) $\sqrt[3]{\sqrt[2]{7}} = \sqrt[3]{7^{\frac{1}{2}}} = \left(7^{\frac{1}{2}}\right)^{\frac{1}{3}} = 7^{\frac{1}{6}} = \sqrt[6]{7} = \sqrt[2\times 3]{7} = \sqrt[2]{\sqrt[3]{7}}$

(iv) $\sqrt[5]{4^3} = \left(4^3\right)^{\frac{1}{5}} = 4^{\frac{3}{5}} = 4^{\frac{9}{15}} = \sqrt[15]{4^9} = \sqrt[3\times 5]{4^{3\times 3}}$

Thus, we see that the above laws of surds are verified.

An important point: The order of a surd can be changed by multiplying the index of the surd and index of the radicand by the same positive number.

For example $\quad \sqrt[3]{2} = \sqrt[6]{2^2} = \sqrt[6]{4}$

and $\quad \sqrt[4]{3} = \sqrt[8]{3^2} = \sqrt[8]{9}$

2.12 SIMILAR (OR LIKE) SURDS

Two surds are said to be **similar**, if they can be reduced to the same irrational factor, without consideration for co-efficient.

For example, $3\sqrt{5}$ and $7\sqrt{5}$ are similar surds. Again consider $\sqrt{75} = 5\sqrt{3}$ and $\sqrt{12} = 2\sqrt{3}$. Now $\sqrt{75}$ and $\sqrt{12}$ are expressed as $5\sqrt{3}$ and $2\sqrt{3}$. Thus, they are similar surds.

2.13 SIMPLEST (LOWEST) FORM OF A SURD

A surd is said to be in its simplest form, if it has

a) smallest possible index of the sign

b) no fraction under radical sign

c) no factor of the form a^n, where a is a positive integer, under the radical sign of index n.

For example, $\sqrt[3]{\dfrac{125}{18}} = \sqrt[3]{\dfrac{125\times 12}{18\times 12}} = \dfrac{5}{6}\sqrt[3]{12}$

Let us take some examples.

Example 2.20: Express each of the following as pure surd in the simplest form:

(i) $2\sqrt{7}$ (ii) $4\sqrt[4]{7}$ (iii) $\dfrac{3}{4}\sqrt{32}$

Solution:

(i) $2\sqrt{7} = \sqrt{2^2 \times 7} = \sqrt{4 \times 7} = \sqrt{28}$, which is a pure surd.

(ii) $4\sqrt[4]{7} = \sqrt[4]{4^4 \times 7} = \sqrt[4]{256 \times 7} = \sqrt[4]{1792}$, which is a pure surd.

(iii) $\dfrac{3}{4}\sqrt{32} = \sqrt{32 \times \dfrac{9}{16}} = \sqrt{18}$, which is a pure surd.

Example 2.21: Express as a mixed surd in the simplest form:

(i) $\sqrt{128}$ (ii) $\sqrt[6]{320}$ (iii) $\sqrt[3]{250}$

Solution:

(i) $\sqrt{128} = \sqrt{64 \times 2} = 8\sqrt{2}$,

which is a mixed surd.

(ii) $\sqrt[6]{320} = 6\sqrt{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 5}$

$= \sqrt[6]{2^6 \times 5} = 2\sqrt[6]{5}$, which is a mixed surd.

(iii) $\sqrt[3]{250} = \sqrt[3]{5 \times 5 \times 5 \times 2} = 5\sqrt[3]{2}$, which is a mixed surd.

CHECK YOUR PROGRESS 2.7

1. State which of the following are pairs of similar surds:

   (i) $\sqrt{8}, \sqrt{32}$ (ii) $5\sqrt{3}, 6\sqrt{18}$ (iii) $\sqrt{20}, \sqrt{125}$

2. Express as a pure surd:

   (i) $7\sqrt{3}$ (ii) $3\sqrt[3]{16}$ (iii) $\dfrac{5}{8}\sqrt{24}$

3. Express as a mixed surd in the simplest form:

   (i) $\sqrt[3]{250}$ (ii) $\sqrt[3]{243}$ (iii) $\sqrt[4]{512}$

2.14 FOUR FUNDAMENTAL OPERATIONS ON SURDS

2.14.1 Addition and Subtraction of Surds

As in rational numbers, surds are added and subtracted in the same way.

For example, $5\sqrt{3}+17\sqrt{3}=(5+17)\sqrt{3}=22\sqrt{3}$

and $12\sqrt{5}-7\sqrt{5}=[12-7]\sqrt{5}=5\sqrt{5}$

For adding and subtracting surds, we first change them to similar surds and then perform the operations.

For example i) $\sqrt{50}+\sqrt{288}$

$= \sqrt{5\times5\times2}+\sqrt{12\times12\times2}$

$= 5\sqrt{2}+12\sqrt{2}=\sqrt{2}(5+12)=17\sqrt{2}$

ii) $\sqrt{98}-\sqrt{18}$

$= \sqrt{7\times7\times2}-\sqrt{3\times3\times2}$

$= 7\sqrt{2}-3\sqrt{2}=(7-3)\sqrt{2}=4\sqrt{2}$

Example 2.22: Simplify each of the following:

(i) $4\sqrt{6}+2\sqrt{54}$

(ii) $45\sqrt{6}-3\sqrt{216}$

Solution: (i) $4\sqrt{6}+2\sqrt{54}$

$= 4\sqrt{6}+2\sqrt{3\times3\times6}$

$= 4\sqrt{6}+6\sqrt{6}=10\sqrt{6}$

(ii) $45\sqrt{6}-3\sqrt{216}$

$= 45\sqrt{6}-3\sqrt{6\times6\times6}$

$= 45\sqrt{6}-18\sqrt{6}$

$= 27\sqrt{6}$

Example 2.23: Show that

$$24\sqrt{45}-16\sqrt{20}+\sqrt{245}-47\sqrt{5}=0$$

Solution: $24\sqrt{45}-16\sqrt{20}+\sqrt{245}-47\sqrt{5}$

$$= 24\sqrt{3\times3\times5} - 16\sqrt{2\times2\times5} + \sqrt{7\times7\times5} - 47\sqrt{5}$$

$$= 72\sqrt{5} - 32\sqrt{5} + 7\sqrt{5} - 47\sqrt{5}$$

$$= \sqrt{5}[72 - 32 + 7 - 47]$$

$$= \sqrt{5}\times 0 = 0 = \text{RHS}$$

Example 2.24: Simplify: $2\sqrt[3]{16000} + 8\sqrt[3]{128} - 3\sqrt[3]{54} + \sqrt[4]{32}$

Solution: $2\sqrt[3]{16000} = 2\sqrt[3]{10\times10\times10\times8\times2} = 2\times10\times2\sqrt[3]{2} = 40\sqrt[3]{2}$

$$8\sqrt[3]{128} = 8\sqrt[3]{4\times4\times4\times2} = 32\sqrt[3]{2}$$

$$3\sqrt[3]{54} = 3\sqrt[3]{3\times3\times3\times2} = 9\sqrt[3]{2}$$

$$\sqrt[4]{32} = 2\sqrt[4]{2}$$

$\therefore$ Required expression

$$= 40\sqrt[3]{2} + 32\sqrt[3]{2} - 9\sqrt[3]{2} + 2\sqrt[4]{2}$$
$$= (40 + 32 - 9)\sqrt[3]{2} + 2\sqrt[4]{2}$$
$$= 63\sqrt[3]{2} + 2\sqrt[4]{2}$$

CHECK YOUR PROGRESS 2.8

Simplify each of the following:

1. $\sqrt{175} + \sqrt{112}$
2. $\sqrt{32} + \sqrt{200} + \sqrt{128}$
3. $3\sqrt{50} + 4\sqrt{18}$
4. $\sqrt{108} - \sqrt{75}$
5. $\sqrt[3]{24} + \sqrt[3]{81} - 8\sqrt[3]{3}$
6. $6\sqrt[3]{54} - 2\sqrt[3]{16} + 4\sqrt[3]{128}$
7. $12\sqrt{18} + 6\sqrt{20} - 6\sqrt{147} + 3\sqrt{50} + 8\sqrt{45}$

2.14.2 Multiplication and Division in Surds

Two surds can be multiplied or divided if they are of the same order. We have read that the order of a surd can be changed by multiplying or dividing the index of the surd and index of the radicand by the same positive number. Before multiplying or dividing, we change them to the surds of the same order.

Let us take some examples:

$$\sqrt{3}\times\sqrt{2}=\sqrt{3\times2}=\sqrt{6} \qquad \left[\sqrt{3} \text{ and } \sqrt{2} \text{ are of same order}\right]$$

$$\sqrt{12}\div\sqrt{2}=\frac{\sqrt{12}}{\sqrt{2}}=\sqrt{6}$$

Let us multiply $\sqrt{3}$ and $\sqrt[3]{2}$

$$\sqrt{3}=\sqrt[6]{3^3}=\sqrt[6]{27}$$

$$\sqrt[3]{2}=\sqrt[6]{4}$$

$$\therefore \sqrt{3}\times\sqrt[3]{2}=\sqrt[6]{27}\times\sqrt[6]{4}=\sqrt[6]{108}$$

$$\text{and } \frac{\sqrt{3}}{\sqrt[3]{2}}=\frac{\sqrt[6]{27}}{\sqrt[6]{4}}=\sqrt[6]{\frac{27}{4}}$$

Let us consider an example:

Example 2.25: (i) Multiply $5\sqrt[3]{16}$ and $11\sqrt[3]{40}$.

(ii) Divide $15\sqrt[3]{13}$ by $6\sqrt[6]{5}$.

Solution: (i) $5\sqrt[3]{16} \times 11\sqrt[3]{40}$

$$= 5\times11\times\sqrt[3]{2\times2\times2\times2}\times\sqrt[3]{2\times2\times2\times5}$$

$$= 55\times2\times2\sqrt[3]{2}\ \sqrt[3]{5}$$

$$= 220\sqrt[3]{10}$$

(ii) $$\frac{15\sqrt[3]{13}}{6\sqrt[6]{5}}=\frac{5}{2}\cdot\frac{\sqrt[6]{13^2}}{\sqrt[6]{5}}=\frac{5}{2}\sqrt[6]{\frac{169}{5}}$$

Example 2.26: Simplify and express the result in simplest form:

$$2\sqrt{50}\times\sqrt{32}\times2\sqrt{72}$$

Solution: $2\sqrt{50} = 2\sqrt{5\times5\times2} = 10\sqrt{2}$

$$\sqrt{32} = \sqrt{2\times2\times2\times2\times2} = 4\sqrt{2}$$

$$2\sqrt{72} = 2\times6\sqrt{2} = 12\sqrt{2}$$

$\therefore$ Given expression

$$= 10\sqrt{2}\times4\sqrt{2}\times12\sqrt{2}$$

$$= 960\sqrt{2}$$

2.15 COMPARISON OF SURDS

To compare two surds, we first change them to surds of the same order and then compare their radicands along with their co-efficients. Let us take some examples:

Example 2.27: Which is greater $\sqrt{\frac{1}{4}}$ or $\sqrt[3]{\frac{1}{3}}$?

Solution: $\sqrt{\frac{1}{4}} = \sqrt[6]{\left(\frac{1}{4}\right)^3} = \sqrt[6]{\frac{1}{64}}$

$$\sqrt[3]{\frac{1}{3}} = \sqrt[6]{\frac{1}{9}}$$

$$\frac{1}{9} > \frac{1}{64} \Rightarrow \sqrt[6]{\frac{1}{9}} > \sqrt[6]{\frac{1}{64}} \Rightarrow \sqrt[3]{\frac{1}{3}} > \sqrt{\frac{1}{4}}$$

Example 2.28: Arrange in ascending order: $\sqrt[3]{2}$, $\sqrt{3}$ and $\sqrt[6]{5}$.

Solution: LCM of 2, 3, and 6 is 6.

$$\therefore \sqrt[3]{2} = \sqrt[6]{2^2} = \sqrt[6]{4}$$

$$\sqrt{3} = \sqrt[6]{3^3} = \sqrt[6]{27}$$

$$\sqrt[6]{5} = \sqrt[6]{5}$$

Now $\sqrt[6]{4} < \sqrt[6]{5} < \sqrt[6]{27}$

$$\Rightarrow \sqrt[3]{2} < \sqrt[6]{5} < \sqrt{3}$$

CHECK YOUR PROGRESS 2.9

1. Multipliy $\sqrt[3]{32}$ and $5\sqrt[3]{4}$.
2. Multipliy $\sqrt{3}$ and $\sqrt[3]{5}$.
3. Divide $\sqrt[3]{135}$ by $\sqrt[3]{5}$.
4. Divide $2\sqrt{24}$ by $\sqrt[3]{320}$.
5. Which is greater $\sqrt[4]{5}$ or $\sqrt[3]{4}$?
6. Which in smaller: $\sqrt[5]{10}$ or $\sqrt[4]{9}$?
7. Arrange in ascending order:

   $\sqrt[3]{2},\ \sqrt[6]{3},\ \sqrt[3]{4}$

8. Arrange in descending order:

   $\sqrt[3]{2},\ \sqrt[4]{3},\ \sqrt[3]{4}$

2.16 RATIONALISATION OF SURDS

Consider the products:

(i) $3^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 3$

(ii) $5^{\frac{7}{11}} \times 5^{\frac{4}{11}} = 5$

(iii) $7^{\frac{1}{4}} \times 7^{\frac{3}{4}} = 7$

In each of the above three multiplications, we see that on multiplying two surds, we get the result as rational number. In such cases, each surd is called the rationalising factor of the other surd.

(i) $\sqrt{3}$ is a rationalising factor of $\sqrt{3}$ and vice-versa.

(ii) $\sqrt[11]{5^4}$ is a rationalising factor of $\sqrt[11]{5^7}$ and vice-versa.

(iii) $\sqrt[4]{7}$ is a rationalising factor of $\sqrt[4]{7^3}$ and vice-versa.

In other words, the process of converting surds to rational numbers is called rationalisation and two numbers which on multiplication give the rational number is called the rationalisation factor of the other.

For example, the rationalising factor of $\sqrt{x}$ is $\sqrt{x}$, of $\sqrt{3}+\sqrt{2}$ is $\sqrt{3}-\sqrt{2}$.

Note:

(i) The quantities $x-\sqrt{y}$ and $x+\sqrt{y}$ are called conjugate surds. Their sum and product are always rational.

(ii) Rationalisation is usually done of the denominator of an expression involving irrational surds.

Let us consider some examples.

Example 2.29: Find the rationalising factors of $\sqrt{18}$ and $\sqrt{12}$.

Solution: $\sqrt{18} = \sqrt{3\times3\times2} = 3\sqrt{2}$

$\therefore$ Rationalising factor is $\sqrt{2}$.

$\sqrt{12} = \sqrt{2\times2\times3} = 2\sqrt{3}$.

$\therefore$ Rationalising factor is $\sqrt{3}$.

Example 2.30: Rationalise the denominator of $\dfrac{\sqrt{2}+\sqrt{5}}{\sqrt{2}-\sqrt{5}}$.

Solution:
$$\frac{\sqrt{2}+\sqrt{5}}{\sqrt{2}-\sqrt{5}} = \frac{\left(\sqrt{2}+\sqrt{5}\right)\left(\sqrt{2}+\sqrt{5}\right)}{\left(\sqrt{2}-\sqrt{5}\right)\left(\sqrt{2}+\sqrt{5}\right)} = \frac{\left(\sqrt{2}+\sqrt{5}\right)^2}{-3}$$

$$= -\frac{7+2\sqrt{10}}{3} = -\frac{7}{3}-\frac{2}{3}\sqrt{10}$$

Example 2.31: Rationalise the denominator of $\dfrac{4+3\sqrt{5}}{4-3\sqrt{5}}$.

Solution:
$$\frac{4+3\sqrt{5}}{4-3\sqrt{5}} = \frac{\left(4+3\sqrt{5}\right)\left(4+3\sqrt{5}\right)}{\left(4-3\sqrt{5}\right)\left(4+3\sqrt{5}\right)}$$

$$= \frac{16+45+24\sqrt{5}}{16-45} = -\frac{61}{29}-\frac{24}{29}\sqrt{5}$$

Example 2.32: Rationalise the denominator of $\dfrac{1}{\sqrt{3}-\sqrt{2}+1}$.

Solution: $\dfrac{1}{\sqrt{3}-\sqrt{2}+1}=\dfrac{\left(\sqrt{3}-\sqrt{2}\right)-1}{\left[\left(\sqrt{3}-\sqrt{2}\right)+1\right]\left[\left(\sqrt{3}-\sqrt{2}\right)-1\right]}$

$$=\frac{\sqrt{3}-\sqrt{2}-1}{\left(\sqrt{3}-\sqrt{2}\right)^2-1}=\frac{\sqrt{3}-\sqrt{2}-1}{4-2\sqrt{6}}$$

$$=\frac{\sqrt{3}-\sqrt{2}-1}{4-2\sqrt{6}}\times\frac{4+2\sqrt{6}}{4+2\sqrt{6}}$$

$$=\frac{4\sqrt{3}-4\sqrt{2}-4+6\sqrt{2}-4\sqrt{3}-2\sqrt{6}}{16-24}$$

$$=-\frac{\sqrt{2}-2-\sqrt{6}}{4}=\frac{\sqrt{6}-\sqrt{2}+2}{4}$$

Example 2.33: If $\dfrac{3+2\sqrt{2}}{3-\sqrt{2}}=a+b\sqrt{2}$, find the values of a and b.

Solution: $\dfrac{3+2\sqrt{2}}{3-\sqrt{2}}=\dfrac{3+2\sqrt{2}}{3-\sqrt{2}}\times\dfrac{3+\sqrt{2}}{3+\sqrt{2}}=\dfrac{9+4+9\sqrt{2}}{9-2}$

$$=\frac{13+9\sqrt{2}}{7}=\frac{13}{7}+\frac{9}{7}\sqrt{2}=\mathrm{a}+\mathrm{b}\sqrt{2}$$

$$\Rightarrow \mathrm{a}=\frac{13}{7},\qquad \mathrm{b}=\frac{9}{7}$$

CHECK YOUR PROGRESS 2.10

1. Find the rationalising factor of each of the following:

   (i) $\sqrt[3]{49}$ (ii) $\sqrt{2}+1$ (iii) $\sqrt[3]{x^2}+\sqrt[3]{y^2}+\sqrt[3]{xy}$

2. Simplify by rationalising the denominator of each of the following:

   (i) $\dfrac{12}{\sqrt{5}}$ (ii) $\dfrac{2\sqrt{3}}{\sqrt{17}}$ (iii) $\dfrac{\sqrt{11}-\sqrt{5}}{\sqrt{11}+\sqrt{5}}$ (iv) $\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$

3. Simplify: $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}+\dfrac{2-\sqrt{3}}{2+\sqrt{3}}$

4. Rationalise the denominator of $\dfrac{1}{\sqrt{3}-\sqrt{2}-1}$

5. If $a=3+2\sqrt{2}$. Find $a+\dfrac{1}{a}$.

6. If $\dfrac{2+5\sqrt{7}}{2-5\sqrt{7}}=x+\sqrt{7}y$, find x and y.

LET US SUM UP

- $a \times a \times a \times \ldots.. m$ times $= a^m$ is the exponential form, where a is the base and m is the exponent.
- Laws of exponent are:

  (i) $a^m \times a^n = a^{m+n}$ (ii) $a^m \div a^n = a^{m-n}$ (iii) $(ab)^m = a^m b^m$ (iv) $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$

  (v) $\left(a^m\right)^n = a^{mn}$ (vi) $a^0 = 1$ (vii) $a^{-m} = \dfrac{1}{a^m}$

- $a^{\frac{p}{q}} = \sqrt[q]{a^p}$
- An irrational number $\sqrt[n]{x}$ is called a surd, if x is a rational number and nth root of x is not a rational number.
- In $\sqrt[n]{x}$, n is called index and x is called radicand.
- A surd with rational co-efficient (other than 1) is called a mixed surd.
- The order of the surd is the number that indicates the root.
- The order of $\sqrt[n]{x}$ is n
- Laws of radicals (a > 0, b > 0)

  (i) $\left[\sqrt[n]{a}\right]^n = a$ (ii) $\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab}$ (iii) $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\dfrac{a}{b}}$

- Operations on surds

$$x^{\frac{1}{n}} \times y^{\frac{1}{n}} = (xy)^{\frac{1}{n}}; \quad \left(x^{\frac{1}{n}}\right)^{\frac{1}{m}} = x^{\frac{1}{mn}} = \left(x^{\frac{1}{m}}\right)^{\frac{1}{n}}; \quad \frac{x^{\frac{1}{n}}}{y^{\frac{1}{n}}} = \left(\frac{x}{y}\right)^{\frac{1}{n}}$$

$$\left(x^{m}\right)^{\frac{1}{n}} = x^{\frac{m}{n}}; \quad \sqrt[m]{x^{a}} = \sqrt[mn]{x^{an}} \text{ or } \left(x^{a}\right)^{\frac{1}{m}} = x^{\frac{a}{m}} = x^{\frac{an}{mn}} = \left(x^{an}\right)^{\frac{1}{mn}}$$

- Surds are similar if they have the same irrational factor.
- Similar surds can be added and subtracted.
- Orders of surds can be changed by multiplying index of the surds and index of the radicand by the same positive number.
- Surds of the same order are multiplied and divided.
- To compare surds, we change surds to surds of the same order. Then they are compared by their radicands alongwith co-efficients.
- If the product of two surds is rational, each is called the rationalising factor of the other.
- $x+\sqrt{y}$ is called rationalising factor of $x-\sqrt{y}$ and vice-versa.

TERMINAL EXERCISE

1. Express the following in exponential form:

   (i) $5 \times 3 \times 5 \times 3 \times 7 \times 7 \times 7 \times 9 \times 9$

   (ii) $\left(\frac{-7}{9}\right)\times\left(\frac{-7}{9}\right)\times\left(\frac{-7}{9}\right)\times\left(\frac{-7}{9}\right)$

2. Simplify the following:

   (i) $\left(-\frac{5}{6}\right)^{3}\times\left(\frac{7}{5}\right)^{2}\times\left(\frac{3}{7}\right)^{3}$

   (ii) $\left(\frac{3}{7}\right)^{2}\times\frac{35}{27}\times\left(-\frac{1}{5}\right)^{2}$

3. Simplify and express the result in exponential form:

   (i) $(10)^{2}\times(6)^{2}\times(5)^{2}$

(ii) $\left(-\frac{37}{19}\right)^{20} \div \left(-\frac{37}{19}\right)^{20}$

(iii) $\left[\left(\frac{3}{13}\right)^{3}\right]^{5}$

4. Simplify each of the following:

   (i) $3^o + 7^o + 37^o - 3$ (ii) $(7^o + 3^o)(7^o - 3^o)$

5. Simplify the following:

   (i) $(32)^{12} \div (32)^{-6}$ (ii) $(111)^{6} \times (111)^{-5}$ (iii) $\left(-\frac{2}{9}\right)^{-3} \times \left(-\frac{2}{9}\right)^{5}$

6. Find x so that $\left(\frac{3}{7}\right)^{-3} \times \left(\frac{3}{7}\right)^{11} = \left(\frac{3}{7}\right)^{x}$

7. Find x so that $\left(\frac{3}{13}\right)^{-2} \times \left(\frac{3}{13}\right)^{-9} = \left(\frac{3}{13}\right)^{2x+1}$

8. Express as a product of primes and write the answers of each of the following in exponential form:

   (i) 6480000 (ii) 172872 (iii) 11863800

9. The star sirus is about 8.1×10^{13} km from the earth. Assuming that the light travels at 3.0×10^{5} km per second, find how long light from sirus takes to reach earth.

10. State which of the following are surds:

    (i) $\sqrt{\frac{36}{289}}$ (ii) $\sqrt[9]{729}$ (iii) $\sqrt[3]{\sqrt{5}+1}$ (iv) $\sqrt[4]{3125}$

11. Express as a pure surd:

    (i) $3\sqrt[2]{3}$ (ii) $5\sqrt[3]{4}$ (iii) $5\sqrt[5]{2}$

12. Express as a mixed surd in simplest form:

    (i) $\sqrt[4]{405}$ (ii) $\sqrt[5]{320}$ (iii) $\sqrt[3]{128}$

13. Which of the following are pairs of similar surds?

    (i) $\sqrt{112}, \sqrt{343}$ (ii) $\sqrt[3]{625}, \sqrt[3]{3125 \times 25}$ (iii) $\sqrt[6]{216}, \sqrt{250}$

14. Simplify each of the following:

(i) $4\sqrt{48}-\frac{5}{2}\sqrt{\frac{1}{3}}+6\sqrt{3}$

(ii) $\sqrt{63}+\sqrt{28}-\sqrt{175}$

(iii) $\sqrt{8}+\sqrt{128}-\sqrt{50}$

15. Which is greater?

(i) $\sqrt{2}$ or $\sqrt[3]{3}$ (ii) $\sqrt[3]{6}$ or $\sqrt[4]{8}$

16. Arrange in descending order:

(i) $\sqrt{3},\sqrt[3]{4},\sqrt[4]{5}$ (ii) $\sqrt{2},\sqrt{3},\sqrt[3]{4}$

17. Arrange in ascending order:

$\sqrt[3]{16},\sqrt{12},\sqrt[6]{320}$

18. Simplify by rationalising the denominator:

(i) $\frac{3}{\sqrt{6}-\sqrt{7}}$ (ii) $\frac{12}{\sqrt{7}-\sqrt{3}}$ (iii) $\frac{\sqrt{5}-2}{\sqrt{5}+2}$

19. Simplify each of the following by rationalising the denominator:

(i) $\frac{1}{1+\sqrt{2}-\sqrt{3}}$ (ii) $\frac{1}{\sqrt{7}+\sqrt{5}-\sqrt{12}}$

20. If $\frac{5+2\sqrt{3}}{7+4\sqrt{3}}=a+b\sqrt{3},$ find the values of a and b, where a and b are rational numbers.

21. If $x=7+4\sqrt{3}$, find the value of $x+\frac{1}{x}$.

ANSWERS TO CHECK YOUR PROGRESS

2.1

1. (i) $(-7)^4$ (ii) $\left(\frac{3}{4}\right)^{10}$ (iii) $\left(\frac{-5}{7}\right)^{20}$

2. | | Base | Exponent |
|---|---|---|
| (i) | -3 | 5 |
| (ii) | 7 | 4 |
| (iii) | $-\frac{2}{11}$ | 8 |

3. (i) $\frac{81}{2401}$ (ii) $\frac{16}{6561}$ (iii) $-\frac{27}{64}$

4. (i) $\frac{3}{7}$ (ii) $\frac{625}{324}$

5. (i) $\left(\frac{1}{3}\right)^5$ (ii) $\left(-\frac{1}{7}\right)^4$ (iii) $\left(-\frac{5}{3}\right)^4$

2.2

1. (i) $3^1 \times 11^1 \times 13^1$ (ii) $2^3 \times 3^4$ (iii) $2^3 \times 3^3 \times 7^1$

2. (i) 3^6 (ii) 2^9 (iii) $2^5 \times 3^4$

   (iv) $\frac{11^3}{2^{12}}$ (v) $\frac{(-7)^3}{2^5}$

2.3

1. (i) $(7)^5$ (ii) $\left(\frac{3}{4}\right)^5$ (iii) $\left(-\frac{7}{8}\right)^6$

2. (i) $(-7)^2$ (ii) $\left(\frac{3}{4}\right)^6$ (iii) $\left(-\frac{7}{8}\right)^{15}$

3. (i) 2^{18} (ii) $\left(\frac{3}{4}\right)^6$ (iii) $\left(-\frac{5}{9}\right)^{15}$

   (iv) $\left(\frac{11}{3}\right)^5$ (v) $\left(-\frac{7}{11}\right)^3$

4. True: (i), (ii), (vii)

   False: (iii), (iv), (v), (vi)

2.4

1. $\frac{49}{9}$

2. (i) $\left(\frac{7}{3}\right)^4$ (ii) 12^2 (iii) $\left(\frac{13}{3}\right)^{12}$

3. (i) $\left(\frac{7}{3}\right)^{-4}$ (ii) $\left(\frac{1}{7}\right)^{-10}$ (iii) $\left(-\frac{4}{3}\right)^{-10}$

4. (i) $\frac{81}{16}$ (ii) $-\frac{2}{3}$ (iii) $-\frac{343}{125}$

5. True: (ii), (iii), (iv)

2.5

1. (i) 8 (ii) $\frac{25}{9}$

2. (i) 1 (ii) $\frac{7}{8}$ (iii) $\frac{13}{16}$

2.6

1. (i) 4, 64 (ii) 6, 343 (iii) 2, 119

2. (iii), (iv)

3. Pure: (i), (iv)

   Mixed: (ii), (iii)

2.7

1. (i), (iii)

2. (i) $\sqrt{147}$ (ii) $\sqrt[3]{432}$ (iii) $\sqrt{\frac{75}{8}}$

3. (i) $5\sqrt[3]{2}$ (ii) $3\sqrt[3]{9}$ (iii) $4\sqrt[4]{2}$

2.8

1. $9\sqrt{7}$ 2. $22\sqrt{2}$ 3. $27\sqrt{2}$ 4. $\sqrt{3}$

5. $-3\sqrt{3}$ 6. $30\sqrt[3]{2}$ 7. $51\sqrt{2}+36\sqrt{5}-42\sqrt{3}$

2.9

1. $20\sqrt[3]{2}$ 2. $3\sqrt[3]{5}$ 3. 3 4. $\sqrt[6]{\dfrac{216}{25}}$

5. $\sqrt[3]{4}$ 6. $\sqrt[4]{9}$ 7. $\sqrt[6]{3}, \sqrt[3]{2}, \sqrt[3]{4}$ 8. $\sqrt[3]{4}, \sqrt[4]{3}, \sqrt[3]{2}$

2.10

1. (i) $\sqrt[3]{7}$ (ii) $\sqrt{2}-1$ (iii) $\sqrt[3]{x}-\sqrt[3]{y}$

2. (i) $\dfrac{12}{5}\sqrt{5}$ (ii) $\dfrac{2\sqrt{51}}{17}$ (iii) $\dfrac{8}{3}-\dfrac{\sqrt{55}}{3}$ (iv) $2+\sqrt{3}$

3. 14

4. $-\dfrac{1}{4}\left[2+\sqrt{6}+\sqrt{2}\right]$

5. 6

6. $-\dfrac{179}{171}-\dfrac{20\sqrt{7}}{171}$

ANSWERS TO TERMINAL EXERCISE

1. (i) $5^2 \times 3^2 \times 7^3 \times 9^2$ (ii) $\left(-\dfrac{7}{9}\right)^4$

2. (i) $-\dfrac{5}{56}$ (ii) $\dfrac{1}{105}$

3. (i) $2^4 \times 3^2 \times 5^4$ (ii) 1 (iii) $\left(\dfrac{3}{13}\right)^{15}$

4. (i) zero (ii) zero

5. (i) $(32)^{18}$ (ii) 111 (iii) $\left(\dfrac{2}{9}\right)^2$

6. $x = 8$

7. $x = -6$

8. $2^7 \times 3^4 \times 5^4$

9. $3^3 \times 10^7$ seconds

10. (ii), (iii), (iv)

11. (i) $\sqrt[2]{27}$ (ii) $\sqrt[3]{500}$ (iii) $\sqrt[5]{6250}$

12. (i) $3\sqrt[4]{5}$ (ii) $2\sqrt[5]{10}$ (iii) $4\sqrt[3]{2}$

13. (i), (ii)

14. (i) $\frac{127}{6}\sqrt{3}$ (ii) zero (iii) $5\sqrt{2}$

15. (i) $\sqrt[3]{3}$ (ii) $\sqrt[3]{6}$

16. (i) $\sqrt{3}, \sqrt[3]{4}, \sqrt[4]{5}$ (ii) $\sqrt{3}, \sqrt[3]{4}, \sqrt{2}$

17. $\sqrt[3]{16}, \sqrt[6]{320}, \sqrt{12}$

18. (i) $-3\left(\sqrt{6}+\sqrt{7}\right)$ (ii) $3\left(\sqrt{7}+\sqrt{3}\right)$ (ii) $9-4\sqrt{5}$

19. (i) $\frac{2+\sqrt{2}+\sqrt{6}}{4}$ (ii) $\frac{7\sqrt{5}+5\sqrt{7}+2\sqrt{105}}{70}$

20. $a = 11$, $b = -6$

21. 14