17

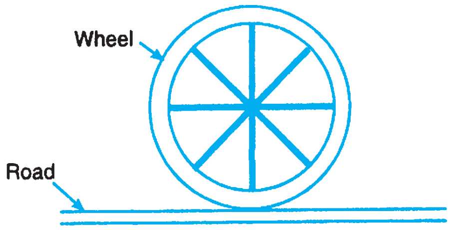
211en17

SECANTS, TANGENTS AND THEIR PROPERTIES

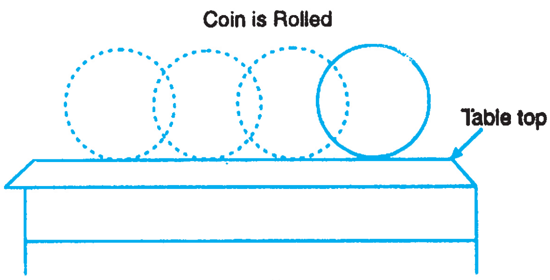
Look at the moving cycle. You will observe that at any instant of time, the wheels of the moving cycle touch the road at a very limited area, more correctly a point.

If you roll a coin on a smooth surface, say a table or floor, you will find that at any instant of time, only one point of the coin comes in contact with the surface it is rolled upon.

What do you observe from the above situations?



(i)



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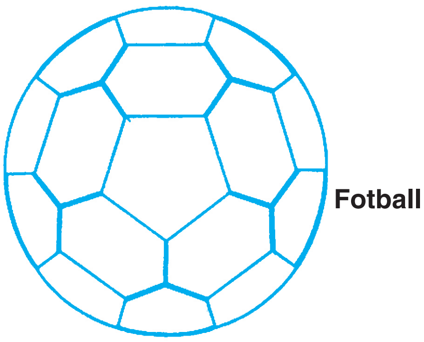

(iii)

Fig. 17.1

If you consider a wheel or a coin as a circle and the touching surface (road or table) as a line, the above illustrations show that a line touches a circle. In this lesson, we shall study about the possible contacts that a line and a circle can have and try to study their properties.

OBJECTIVES

After studying this lesson, you will be able to

- *define a secant and a tangent to the circle;*
- *differentitate between a secant and a tangent;*
- *prove that the tangents drawn from an external point to a circle are of equal length;*
- *verify the un-starred results (given in the curriculum) related to tangents and secants to circle experimentally.*

EXPECTED BACKGROUND KNOWLEDGE

- Measurement of angles and line segments
- Drawing circles of given radii
- Drawing lines perpendicular and parallel to given lines
- Knowledge of previous results about lines and angles, congruence and circles
- Knowledge of Pythagoras Theorem

17.1 SECANTS AND TANGENTS—AN INTRODUCTION

You have read about lines and circles in your earlier lessons. Recall that a circle is the locus of a point in a plane which moves in such a way that its distance from a fixed point in the plane always remains constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle. You also know that a line is a collection of points, extending indefinitely to both sides, whereas a line segment is a portion of a line bounded by two points.

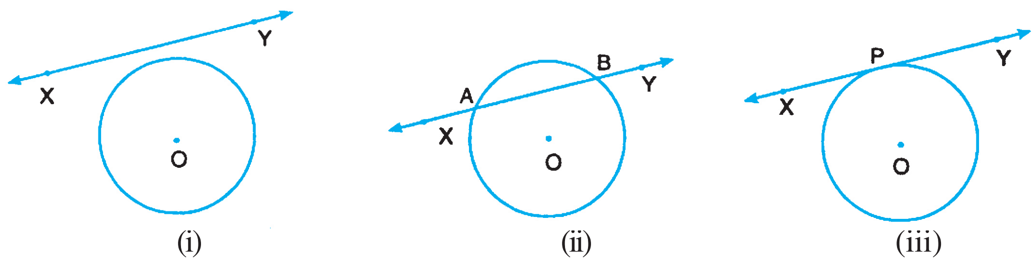

Fig. 17.2

Now consider the case when a line and a circle co-exist in the same plane. There can be three distinct possibilities as shown in Fig. 17.2.

You can see that in Fig. 17.2(i), the XY does not intersect the circle, with centre O. In other words, we say that the line XY and the circle have no common point. In Fig. 17.2 (ii), the line XY intersects the circle in two distinct point A and B, and in Fig. 17.2 (iii), the line XY intersects the circle in only one point and is said to touch the circle at the point P.

Thus, we can say that in case of intersection of a line and a circle, the following three possibilities are there:

(i) The line does not intersect the circle at all, i.e., the line lies in the exterior of the circle.

(ii) The line intersects the circle at two distinct points. In that case, a part of the line lies in the interior of the circle, the two points of intersection lie on the circle and the remaining portion of the line lies in the exterior of the circle.

(iii) The line touches the circle in exactly one point. We therefore define the following:

Tangent:

A line which touches a circle at exactly one point is called a tangent line and the point where it touches the circle is called the point of contact

Thus, in Fig. 17.2 (iii), XY is a tangent of the circle at P, which is called the point of contact.

Secant:

A line which interesects the circle in two distinct points is called a secant line (usually referred to as a secant).

In Fig. 17.2 (ii), XY is a secant line to the circle and A and B are called the points of intersection of the line XY and the circle with centre O.

17.2 TANGENT AS A LIMITING CASE

Consider the secant XY of the circle with centre O, intersecting the circle in the points A and B. Imagine that one point A, which lies on the circle, of the secant XY is fixed and the secant rotates about A, intersecting the circle at B′, B″, B‴, B⁗ as shown in Fig. 17.3 and ultimately attains the position of the line XAY, when it becomes tangent to the circle at A.

Thus, we say that:

A tangent is the limiting position of a secant when the two points of intersection coincide.

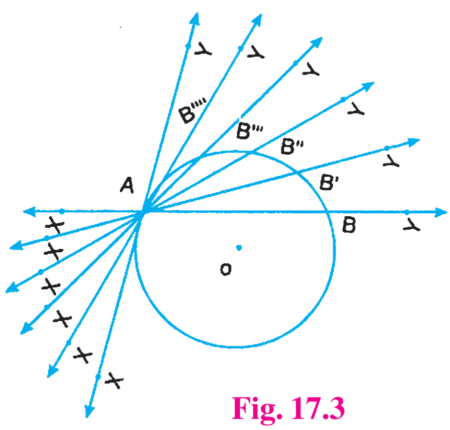

Fig. 17.3

17.3 TANGENT AND RADIUS THROUGH THE POINT OF CONTACT

Let XY be a tangent to the circle, with centre O, at the point P. Join OP.

Take points Q, R, S and T on the tangent XY and join OQ, OR, OS and OT.

As Q, R, S and T are points in the exterior of the circle and P is on the circle.

$\therefore$ OP is less than each of OQ, OR, OS and OT.

From our, "previous study of Geometry, we know that of all the segments that can be drawn from a point (not on the line) to the line, the perpendicular segment is the shortest":

As OP is the shortest distance from O to the line XY

$$\therefore \quad OP \perp XY$$

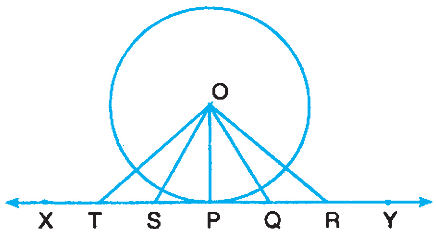

Fig. 17.4

Thus, we can state that

A radius, though the point of contact of tangent to a circle, is perpendicular to the tangent at that point.

The above result can also be verified by measuring angles OPX and OPY and finding each of them equal to 90°.

17.4 TANGENTS FROM A POINT OUTSIDE THE CIRCLE

Take any point P in the exterior of the circle with centre O. Draw lines through P. Some of these are shown as PT, PA, PB, PC, PD and PT′ in Fig. 17.5

How many of these touch the circle? Only two.

Repeat the activity with another point and a circle. You will again find the same result.

Thus, we can say that

From an external point, two tangents can be drawn to a circle.

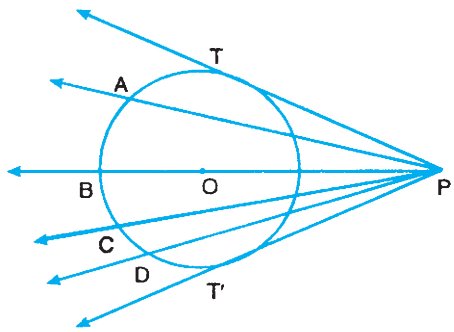

Fig. 17.5

If the point P lies on the circle, can there still be two tangents to the circle from that point? You can see that only one tangent can be drawn to the circle in that case. What about the case when P lies in the interior of the circle? Note that any line through P in that case will intersect the circle in two points and hence no tangent can be drawn from an interior point to the circle.

(A) Now, measure the lengths of PT and PT′. You will find that

$$PT = PT' \qquad \text{....(i)}$$

(B) **Given:** A circle with centre O. PT and PT′ are two tangents from a point P outside the circle.

To Prove: PT = PT′

Construciton: Join OP, OT and OT′ (see Fig. 17.6)

Proof: In Δ's OPT and OPT′

$\angle$ OTP = $\angle$ OT′P (Each being right angle)

OT = OT′

OP = OP (Common)

ΔOPT $\cong$ ΔOPT′ (RHS criterion)

$\therefore$ PT = PT′

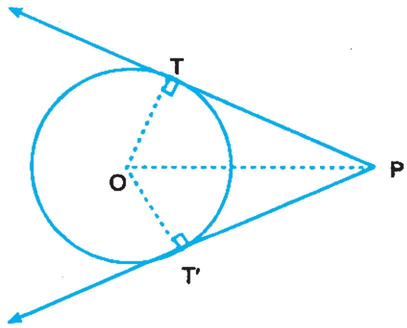

Fig. 17.6

The lengths of two tangents from an external point are equal

Also, from Fig. 17.6, $\angle$ OPT = $\angle$ OPT′ (As ΔOPT $\cong$ ΔOPT′)

The tangents drawn from an external point to a circle are equally inclined to the line joining the point to the centre of the circle.

Let us now take some examples to illustrate:

Example 17.1: In Fig. 17.7, OP = 5 cm and radius of the circle is 3 cm. Find the length of the tangent PT from P to the circle, with centre O.

Solution: $\angle$OTP = 90°, Let PT = x

In right triangle OTP, we have

$OP^2 = OT^2 + PT^2$

or $5^2 = 3^2 + x^2$

or $x^2 = 25 - 9 = 16$

$\therefore$ x = 4

i.e. the length of tangent PT = 4 cm

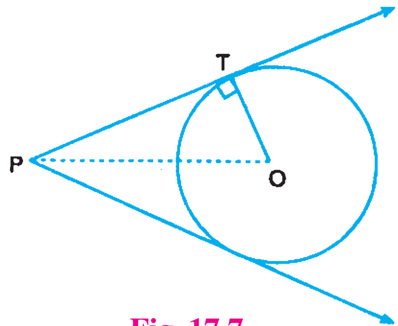

Fig. 17.7

Example 17.2: In Fig. 17.8, tangents PT and PT′ are drawn from a point P at a distance of 25 cm from the centre of the circle whose radius is 7 cm. Find the lengths of PT and PT′.

Solution: Here OP = 25 cm and OT = 7 cm

We also know that

$\angle$OTP = 90°

$\therefore$ $PT^2 = OP^2 - OT^2$

$= 625 - 49 = 576 = (24)^2$

$\therefore$ PT = 24 cm

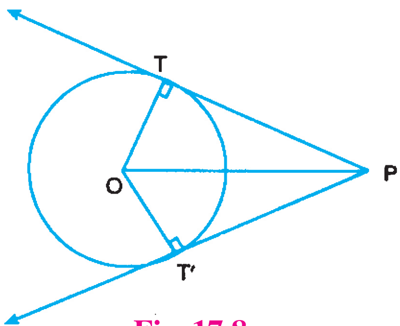

Fig. 17.8

We also know that

$$PT = PT'$$

$\therefore \quad PT' = 24$ cm

Example 17.3: In Fig. 17.9, A, B and C are three exterior points of the circle with centre O. The tangents AP, BQ and CR are of lengths 3 cm, 4 cm and 3.5 cm respectively. Find the perimeter of ΔABC.

Solution: We know that the lengths of two tangents from an external point to a circle are equal

$\therefore \quad AP = AR$

$BP = BQ,$

$CQ = CR$

$\therefore \quad AP = AR = 3$ cm

$BP = BQ = 4$ cm

and $\quad CR = CQ = 3.5$ cm

$AB = AP + PB;$

$= (3 + 4)$ cm $= 7$ cm

$BC = BQ + QC;$

$= (4 + 3.5)$ cm $= 7.5$ cm

$CA = AR + CR$

$= (3 + 3.5)$ cm

$\therefore \quad = 6.5$ cm

$\therefore \quad$ Perimeter of $\Delta ABC = (7 + 7.5 + 6.5)$ cm $= 21$ cm

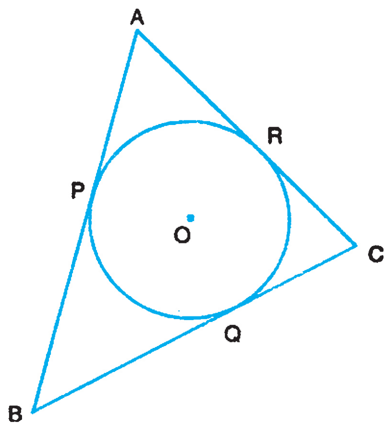

Fig. 17.9

Example 17.4: In Fig. 17.10, $\angle AOB = 50^\circ$. Find $\angle ABO$ and $\angle OBT$.

Solution: We know that $OA \perp XY$

$\Rightarrow \quad \angle OAB = 90^\circ$

$\therefore \quad \angle ABO = 180^\circ - (\angle OAB + \angle AOB)$

$= 180^\circ - (90^\circ + 50^\circ) = 40^\circ$

We know that $\quad \angle OAB = \angle OBT$

$\Rightarrow \quad \angle OBT = 40^\circ$

$\therefore \quad \angle ABO = \angle OBT = 40^\circ$

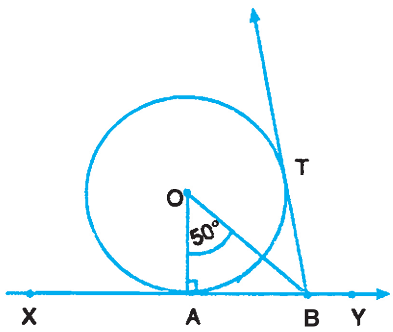

Fig. 17.10

CHECK YOUR PROGRESS 17.1

1. Fill in the blanks:
   (i) A tangent is __________ to the radius through the point of contact.
   (ii) The lengths of tangents from an external point to a circle are __________
   (iii) A tangent is the limiting position of a secant when the two ______ coincide.
   (iv) From an external point ________ tangents can be drawn to a circle.
   (v) From a point in the interior of the circle, ______ tangent(s) can be drawn to the circle.
2. In Fig. 17.11, $\angle POY = 40^o$, Find the $\angle OYP$ and $\angle OYT$.
3. In Fig. 17.12, the incircle of ΔPQR is drawn. If PX = 2.5 cm, RZ = 3.5 cm and perimeter of ΔPQR = 18 cm, find the lenght of QY.

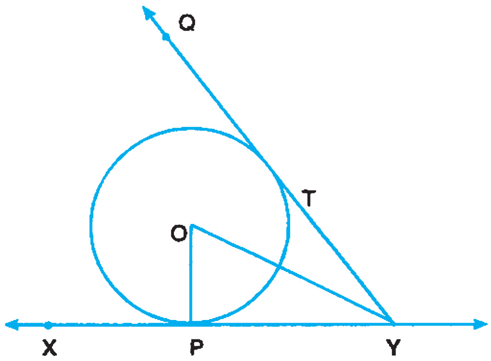

Fig. 17.11

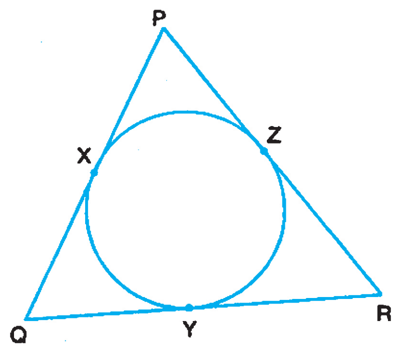

Fig. 17.12

4. Write an experiment to show that the lengths of tangents from an external point to a circle are equal.

17.5 INTERSECTING CHORDS INSIDE AND OUTSIDE A CIRCLE

You have read various results about chords in the previous lessons. We will now verify some results regarding chords intersecting inside a circle or outside a circle, when produced.

Let us perform the following activity:

Draw a circle with centre O and any radius. Draw two chords AB and CD intersecting at P inside the circle.

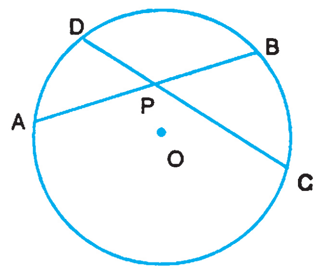

Fig. 17.13

Measure the lenghts of the line-segments PD, PC, PA and PB. Find the products PA × PB and PC × PD.

You will find that they are equal.

Repeat the above activity with another circle after drawing chrods intersecting inside. You will again find that

$$PA \times PB = PC \times PD$$

Let us now consider the case of chrods intersecting outside the circle. Let us perform the following activity:

Draw a circle of any radius and centre O. Draw two chords BA and DC intersecting each other outside the circle at P. Measure the lengths of line segments PA, PB, PC and PD. Find the products PA × PB and PC × PD.

You will see that the product PA × PB is equal to the product PC × PD, i.e.,

$$PA \times PB = PC \times PD$$

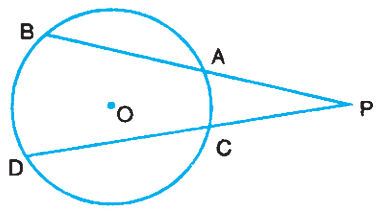

Fig. 17.14

Repeat this activity with two circles with chords intersecting outside the circle. You will again find that

$$PA \times PB = PC \times PD.$$

Thus, we can say that

If two chords AB and CD of a circle intersect at a point P (inside or outside the circle), then

$$\mathbf{PA \times PB = PC \times PD}$$

17.6 INTERSECTING SECANT AND TANGENT OF A CIRCLE

To see if there is some relation beween the intersecting secant and tangent outside a circle, we conduct the following activity.

Draw a circle of any radius with centre O. From an external point P, draw a secant PAB and a tangent PT to the circle.

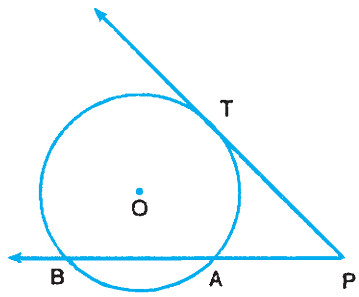

Fig. 17.15

Measure the length of the line-segment PA, PB and PT. Find the products PA × PB and PT × PT or PT^2. What do you find?

You will find that

$$PA \times PB = PT^2$$

Repeat the above activity with two other circles. You will again find the same result.

Thus, we can say

If PAB is a secant to a circle intersecting the circle at A and B, and PT is a tangent to the circle at T, then

$$PA \times PB = PT^2$$

Let us illustrate these with the help of examples:

Example 17.5: In Fig. 17.16, AB and CD are two chords of a circle intersecting at a point P inside the circle. If PA = 3 cm, PB = 2 cm and PC = 1.5 cm, then find the length of PD.

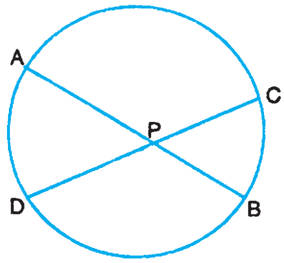

Fig. 17.16

Solution: It is given that PA = 3 cm, PB = 2 cm and PC = 1.5 cm.

Let $\quad PD = x$

We know that $\quad PA \times PB = PC \times PD$

$\Rightarrow \quad 3 \times 2 = (1.5) \times x$

$$\Rightarrow \quad x = \frac{3 \times 2}{1.5} = 4$$

$\therefore$ Length of the line-segment PD = 4 cm.

Example 17.6: In Fig. 17.17, PAB is a secant to the circle from a point P outside the circle. PAB passes through the centre of the circle and PT is a tangent. If PT = 8 cm and OP = 10 cm, find the radius of the circle, using $PA \times PB = PT^2$

Solution: Let x be the radius of the circle.

It is given that $\quad OP = 10$ cm

$\therefore \quad PA = PO - OA = (10 - x)$ cm

and $\quad PB = OP + OB = (10 + x)$ cm

$\quad PT = 8$ cm

We know that $\quad PA \times PB = PT^2$

$\therefore \quad (10 - x)(10 + x) = 8^2$

or $\quad 100 - x^2 = 64$

or $\quad x^2 = 36$ or $x = 6$

i.e., radius of the circle is 6 cm.

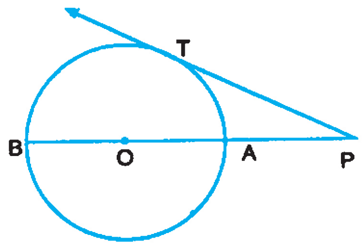

Fig. 17.17

Example 17.7: In Fig. 17.18, BA and DC are two chords of a circle intersecting each other at a point P outside the circle. If PA = 4 cm, PB = 10 cm, CD = 3 cm, find PC.

Solution: We are given that PA = 4 cm, PB = 10 cm, CD = 3 cm

Let PC = x

We know that $PA \times PB = PC \times PD$

or $4 \times 10 = (x + 3)\,x$

or $x^2 + 3x - 40 = 0$

$(x + 8)(x - 5) = 0$

$\Rightarrow \quad x = 5$

$\therefore \quad PC = 5$ cm

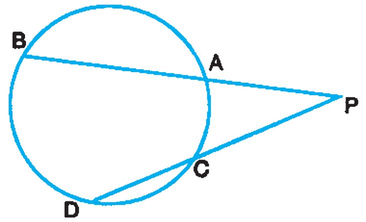

Fig. 17.18

CHECK YOUR PROGRESS 17.2

1. In Fig. 17.19, if PA = 3 cm, PB = 6 cm and PD = 4 cm then find the length of PC.
2. In Fig. 17.19, PA = 4 cm, PB = x + 3, PD = 3 cm and PC = x + 5, find the value of x.

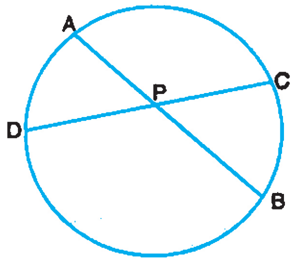

Fig. 17.19

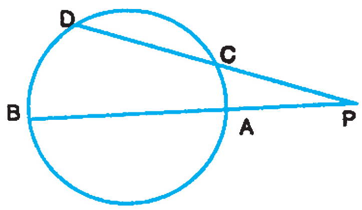

Fig. 17.20

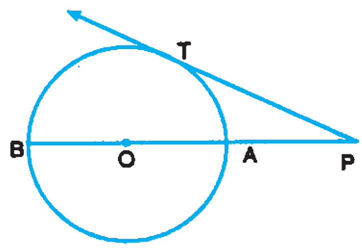

Fig. 17.21

3. If Fig. 17.20, if PA = 4cm, PB = 10 cm, PC = 5 cm, find PD.
4. In Fig. 17.20, if PC = 4 cm, PD = (x + 5) cm, PA = 5 cm and PB = (x + 2) cm, find x.
5. In Fig. 17.21, PT = $2\sqrt{7}$ cm, OP = 8 cm, find the radius of the circle, if O is the centre of the circle.

17.7 ANGLES MADE BY A TANGENT AND A CHORD

Let there be a circle with centre O and let XY be a tangent to the circle at point P. Draw a chord PQ of the circle through the point P as shown in the Fig. 17.22. Mark a point R on the major arc PRQ and let S be a point on the minor arc PSQ.

The segment formed by the major arc PRQ and chord PQ is said to be the alternate segment of ∠QPY and the segment formed by the minor PSQ and chord PQ is said to be the alternate segment to ∠QPX.

Let us see if there is some relationship between angles in the alternate segment and the angle between tangent and chord.

Join QR and PR.

Measure $\angle PRQ$ and $\angle QPY$ (See Fig. 17.22)

What do you find? You will see that $\angle PRQ = \angle QPY$

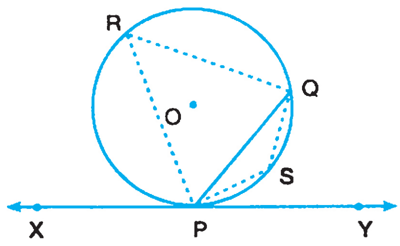

Fig. 17.22

Repeat this activity with another circle and same or different radius. You will again find that $\angle QPY = \angle PRQ$

Now measure $\angle QPX$ and $\angle QSP$. You will again find that these angles are equal.

Thus, we can state that

The angles formed in the alternate segments by a chord through the point of contact of a tangent to a circle is equal to the angle between the chord and the tangent.

This result is more commonly called as "Angles in the Alternate Segment".

Let us now check the converse of the above result.

Draw a circle, with centre O, and draw a chord PQ and let it form $\angle PRQ$ in alternate segment as shown in Fig. 17.23.

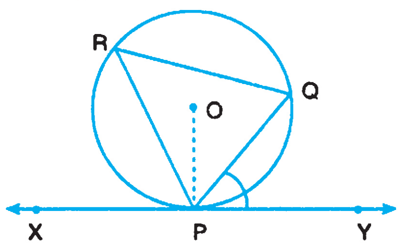

Fig. 17.23

At P, draw $\angle QPY = \angle QRP$. Extend the line segment PY to both sides to form line XY. Join OP and measure $\angle OPY$.

What do you observe? You will find that $\angle OPY = 90^o$ showing thereby that XY is a tangent to the circle.

Repeat this activity by taking different circles and you find the same result. Thus, we can state that

If a line makes with a chord angles which are equal respectively to the angles formed by the chord in alternate segments, then the line is a tangent to the circle.

Let us now take some examples to illustrate:

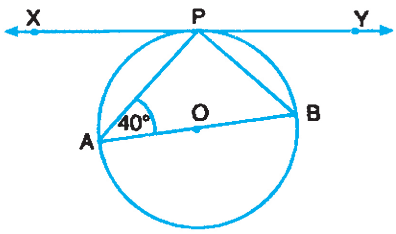

Fig. 17.24

Example 17.8: In Fig. 17.24, XY is tangent to a circle with centre O. If AOB is a diameter and $\angle PAB = 40^o$, find $\angle APX$ and $\angle BPY$.

Solution: By the Alternate Segment theorem, we know that

$\angle BPY = \angle BAP$

$\therefore \quad \angle BPY = 40^\circ$

Again, $\angle APB = 90^\circ$ (Angle in a semi-circle]

And, $\angle BPY + \angle APB + \angle APX = 180^\circ$ (Angles on a line)

$\therefore \quad \angle APX = 180^\circ - (\angle BPY + \angle APB)$

$= 180^\circ - (40^\circ + 90^\circ) = 50^\circ$

Example 17.9: In Fig. 17.25, ABC is an isoceles triangle with AB = AC and XY is a tangent to the circumcircle of ΔABC. Show that XY is parallel to base BC.

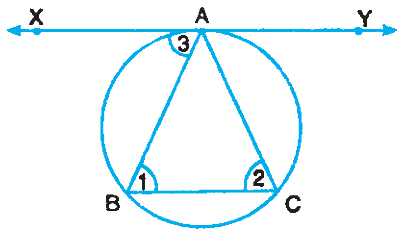

Fig. 17.25

Solution: In ΔABC, AB = AC

$\therefore \quad \angle 1 = \angle 2$

Again XY is tangent to the circle at A.

$\therefore \quad \angle 3 = \angle 2$ (Angles in the alternate segment)

$\therefore \quad \angle 1 = \angle 3$

But these are alternate angles

$\therefore \quad XY \parallel BC$

CHECK YOUR PROGRESS 17.3

1. Explain with the help of a diagram, the angle formed by a chord in the alternate segment of a circle.
2. In Fig. 17.26, XY is a tangent to the circle with centre O at a point P. If $\angle OQP = 40^\circ$, find the value of a and b.

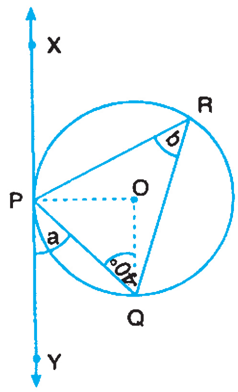

Fig. 17.26

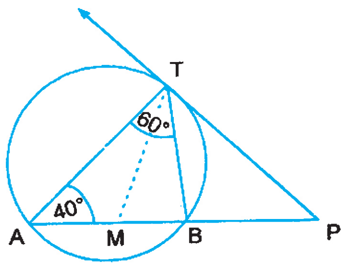

Fig.17.27

3. In Fig. 17.27, PT is a tangent to the circle from an external point P. Chord AB of the circle, when produced meets TP in P. TA and TB are joined and TM is the angle bisector of $\angle ATB$.

   If $\angle PAT = 40^o$ and $\angle ATB = 60^o$, show that PM = PT.

LET US SUM UP

- A line which intersects the circle in two points is called a secant of the circle.
- A line which touches the circle at a point is called a tangent to the circle.
- A tangent is the limiting position of a secant when the two points of intersection coincide.
- A tangent to a circle is perpendicular to the radius through the point of contact.
- From an external point, two tangents can be drawn to a circle, which are of equal length.
- If two chords AB and CD of a circle intersect at a point P (inside or outside the circle), then

  $PA \times PB = PC \times PD$
- If PAB is a secant to a circle intersecting the circle at A and B, and PT is a tangent to the circle at T, then

  $PA \times PB = PT^2$
- The angles formed in the alternate segments by a chord through the point of contact of a tangent to a circle are equal to the angles between the chord and the tangent.
- If a line makes with a chord angles which are respectively equal to the angles formed by the chord in alternate segments, then the line is a tangent to the circle.

TERMINAL EXERCISE

1. Differentiate between a secant and a tangent to a circle with the help of a figure.
2. Show that a tangent is a line perpendicular to the radius through the point of contact, with the help of an activity.
3. In Fig. 17.28, if AC = BC and AB is a diameter of circle, find $\angle x$, $\angle y$ and $\angle z$.

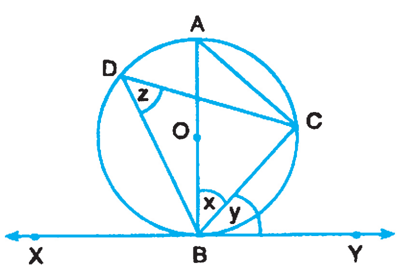

Fig. 17.28

4. In Fig. 17.29, OT = 7 cm and OP = 25 cm, find the length of PT. If PT′ is another tangent to the circle, find the length of PT′ and ∠POT.

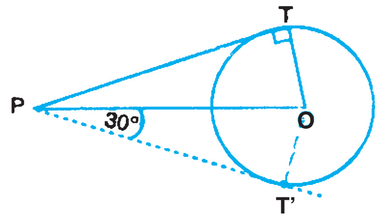

Fig. 17.29

5. In Fig. 17.30, the perimeter of ΔABC equals 27 cm. If PA = 4 cm, QB = 5 cm, find the length of QC.

6. In Fig. 17.30, if ∠ABC = 70°, find ∠BOC.

[Hint: $\angle OBC + \angle OCB = \frac{1}{2}(\angle ABC + \angle ACB)$]

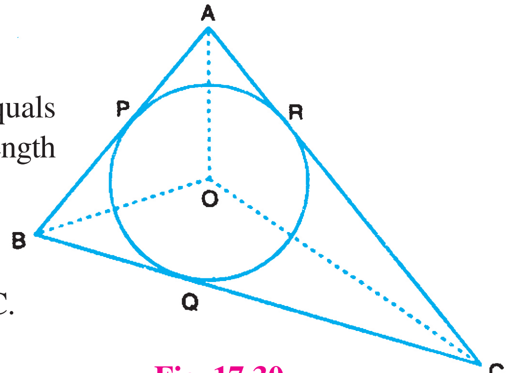

Fig. 17.30

7. In Fig. 17.31, AB and CD are two chords of a circle intersecting at the interior point P of a circle. If PA = (x + 3) cm, PB = (x – 3) cm, PD = 3 cm and PC = $5\frac{1}{3}$ cm, find x.

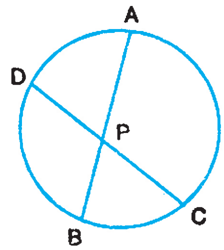

Fig. 17.31

8. In Fig. 17.32, chords BA and DC of the circle, with centre O, intersect at a point P outside the circle. If PA = 4 cm and PB = 9 cm, PC = x and PD = 4x, find the value of x.

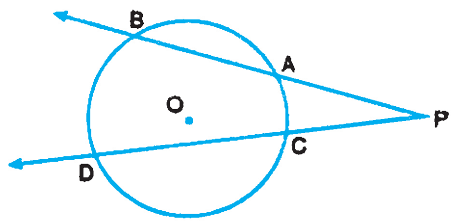

Fig. 17.32

9. In Fig. 17.33, PAB is a secant and PT is a tangent to the circle from an external point. If PT = x cm, PA = 4 cm and AB = 5 cm, find x.

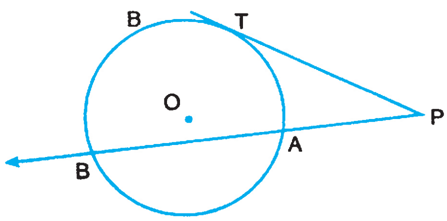

Fig. 17.33

10. In Fig. 17.34, O is the centre of the circle and $\angle PBQ = 40^o$, find

    (i) $\angle QPY$

    (ii) $\angle POQ$

    (iii) $\angle OPQ$

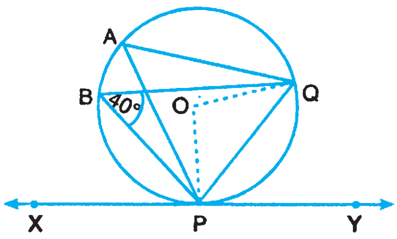

Fig. 17.34

ANSWERS TO CHECK YOUR PROGRESS

17.1

1. (i) Perpendicular (ii) equal (iii) points of intersection

   (iv) two (v) no

2. 50^o, 50^o

3. 3 cm

17.2

1. 4.3 cm 2. 3 cm 3. 8 cm

4. 10 cm 4. 6 cm

17.3

2. $\angle a = \angle b = 50^o$

ANSWERS TO TERMINAL EXERCISE

1. $\angle x = \angle y = \angle z = 45^o$

4. PT = 24 cm; PT' = 24 cm, $\angle POT' = 60^o$

5. QC = 4.5 6. $\angle BOC = 125^o$

7. x = 5 8. x = 3

9. x = 6

10. (i) 40^o (ii) 80^o (iii) 50^o