12

CONCURRENT LINES

You have already learnt about concurrent lines, in the lesson on lines and angles. You have also studied about triangles and some special lines, i.e., medians, right bisectors of sides, angle bisectors and altitudes, which can be drawn in a triangle. In this lesson, we shall study the concurrency property of these lines, which are quite useful.

OBJECTIVES

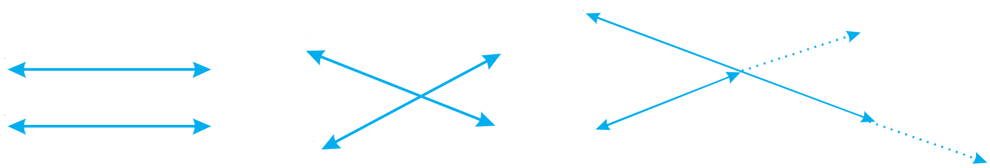
After studying this lesson, you will be able to

- *define the terms concurrent lines, median, altitude, angle bisector and perpendicular bisector of a side of a triangle.*
- *Verify the concurrnence of medians, altitudes, perpendicular bisectors of sides and angle bisectors of a triangle.*

EXPECTED BACKGROUND KNOWLEDGE

Properties of intersecting lines, such as:

- Two lines in a plane can either be parallel [See Fig 12.1 (a)] or intersecting (See Fig. 12.1 (b) and (c)].



(a) (b) (c)

Fig. 12.1

- Three lines in a plane may
  (i) be paralled to each other, i.e., intersect in no point [See Fig. 12.2 (a)] or
  (ii) intersect each other in exactly one point [Fig. 12.2(b)], or
  (iii) intersect each other in two points [Fig. 12.2(c)], or
  (iv) intersect each other at the most in three points [Fig. 12.2(d)]

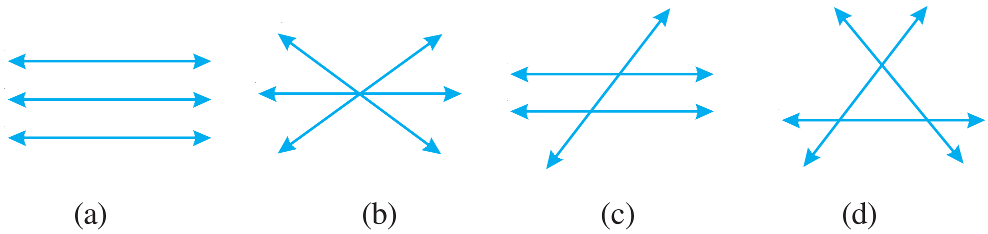


Fig. 12.2

12.1 CONCURRENT LINES

Three lines in a plane which intersect each other in exactly one point or which pass through the same point are called **concurrent lines** and the common **point** is called the **point of concurrency** (See Fig. 12.3).

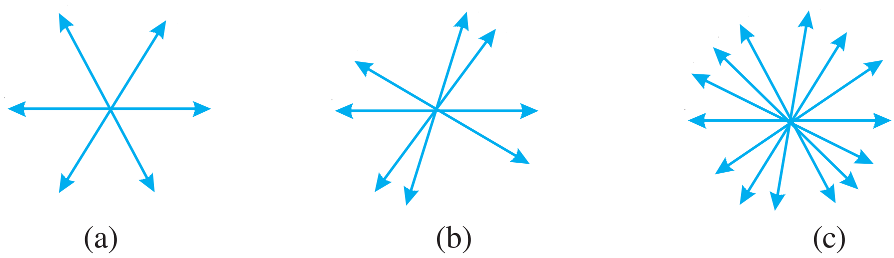


Fig. 12.3

12.1.1 Angle Bisectors of a Triangle

In triangle ABC, the line AD bisects $\angle A$ of the triangle. (See Fig. 12.4)

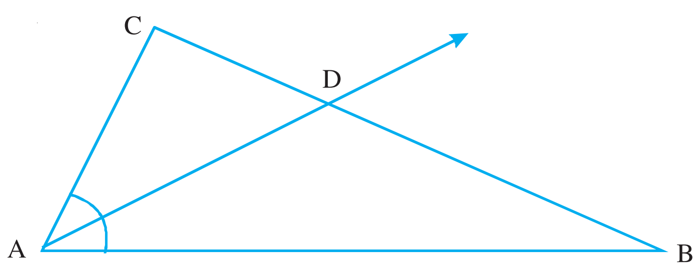


Fig. 12.4

A line which bisects an angle of a triangle is called an **angle bisector** of the triangle.

How many angle bisectors can a triangle have? Since a triangle has three angles, we can draw three angle bisectors in it. AD is one of the three angle bisectors of ΔABC. Let us draw second angle bisector BE of $\angle B$ (See Fig. 12.5)

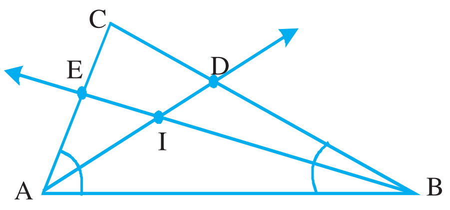


Fig. 12.5

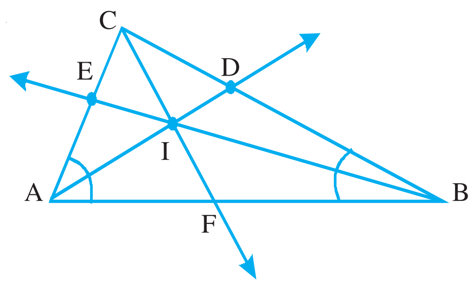


Fig. 12.6

The two angle bisectors of the ΔABC intersect each other at I. Let us draw the third angle bisector CF of $\angle C$ (See Fig. 12.6). We observe that this angle bisector of the triangle also passes through I. In other words they are concurrent and the point of concurrency is I.

We may take any type of triangle—acute, right or obtuse triangle, and draw its angle bisectors, we will always find that the three angle bisectors of a triangle are concurrent (See Fig. 12.7)

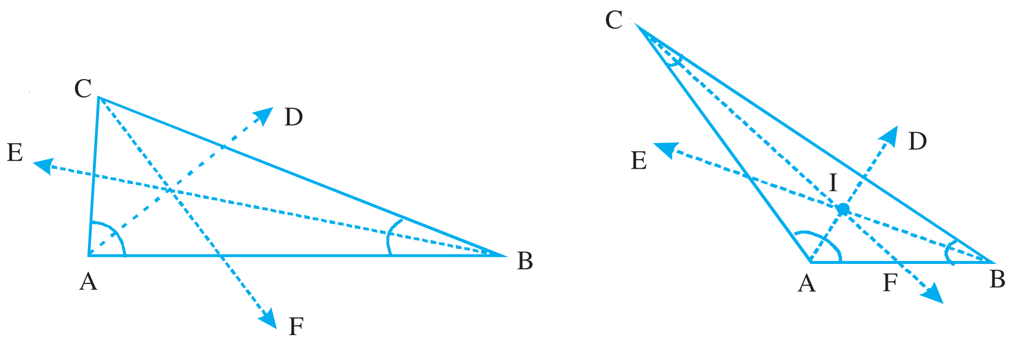


Fig. 12.7

Thus we conclude the following:

Angle bisectors of a triangle pass through the same point, that is they are concurrent

The point of concurrency I is called the 'Incentre' of the triangle.

Can you reason out, why the name incentre for this point?

Recall that the locus of a point equidistant from two intersecting lines is the pair of angle bisectors of the angles formed by the lines. Since I is a point on the bisector of $\angle BAC$, it must be equidistant from AB and AC. Also I is a point on angle bisector of $\angle ABC$, (See

Fig. 12.8), it must also be equidistant from AB and BC. Thus point of concurrency I is at the same distance from all the three sides of the triangle.

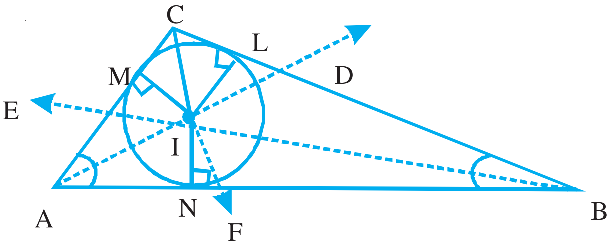


Fig. 12.8

Thus, we have IL = IM = IN (Fig. 12.8). Taking I as the centre and IL as the radius, we can draw a circle touching all the three sides of the triangle called **'Incircle'** of the triangle. I being the centre of the incircle is called the **Incentre** and IL, the radius of the incircle is called the inradius of the triangle.

Note: The incentre always lies in the interior of the triangle.

12.1.2: Perpendicular Bisectors of the Sides of a Triangle

ABC is a triangle, line DP bisects side BC at right angle. A line which bisects a side of a triangle at right angle is called the perpendicular bisector of the side. Since a triangle has three sides, so we can draw three perpendicular bisectors in a triangle. DP is one of the three perpendicular bisectors of ΔABC (Fig. 12.9). We draw the second perpendicular bisector EQ, intersecting DP at O (Fig. 12.10). Now if we also draw the third perpendicular bisector FR, then we observe that it also passes through the point O (Fig. 12.11). In other words, we can say that the three perpendicular bisectors of the sides are concurrent at O.

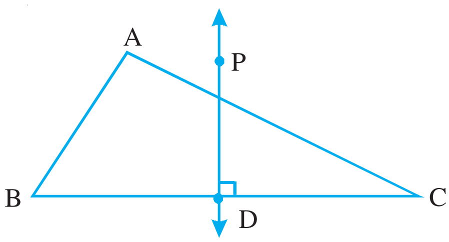


Fig. 12.9

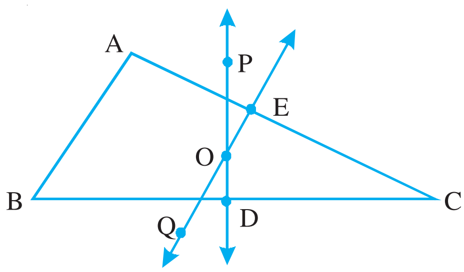


Fig. 12.10

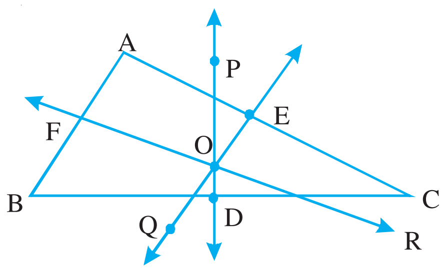


Fig. 12.11

We may repeat this experiment with any type of triangle, but we will always find that the three perpendicular bisectors of the sides of a triangle pass through the same point.

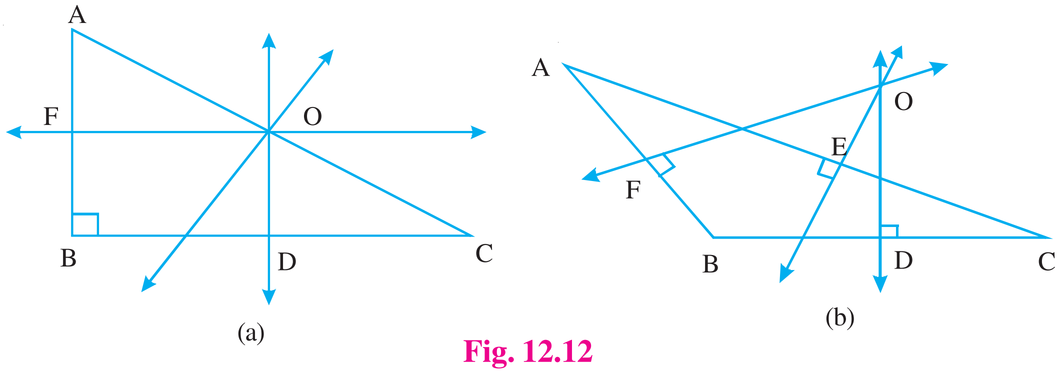


Fig. 12.12

Thus we conclude that:

The three perpendicular bisectors of the sides of a triangle pass through the same point, that is, they are concurrent.

The point of concurrency O is called the 'circumcentre' of the triangle

Can you reason out: why the name circumcentre for this point?

Recall that the locus of a point equidistant from two given points is the perpendicular bisector of the line joining the two points. Since O lies on the perpendicular bisector of BC, so it must be equidistant from both the point B and C i.e., BO = CO (Fig. 12.13).

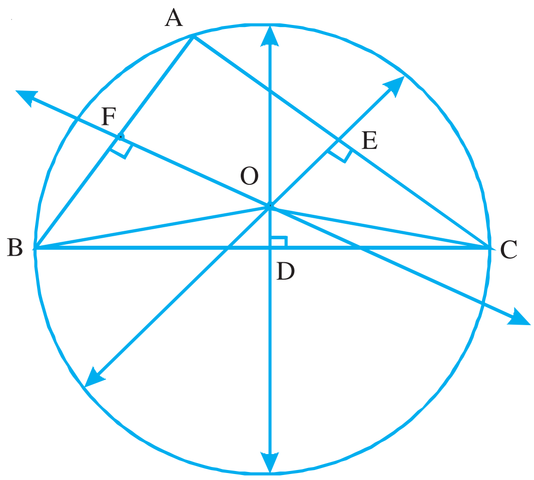


Fig. 12.13

The point O also lies on the perpendicular bisector of AC, so it must be equidistant from both A and C, that is, AO = CO. Thus, we have AO = BO = CO.

If we take O as the centre and AO as the radius, we can draw a circle passing through the three vertices, A, B and C of the triangle, called '**Circumcircle**' of the triangle. O being the centre of this circle is called the **circumcentre** and AO the radius of the circumcircle is called **circumradius** of the triangle.

Note that the circumcentre will be

1. in the interior of the triangle for an acute triangle (Fig. 12.11)
2. on the hypotenuse for a right triangle [Fig. 12.12(a)]
3. in the exterior of the triangle for an obtuse triangle [Fig. 12.12(b)].

12.1.3 Altitudes of a Triangle

In ΔABC, the line AL is the perpendicular drawn from vertex A to the opposite side BC. (Fig. 12.14).

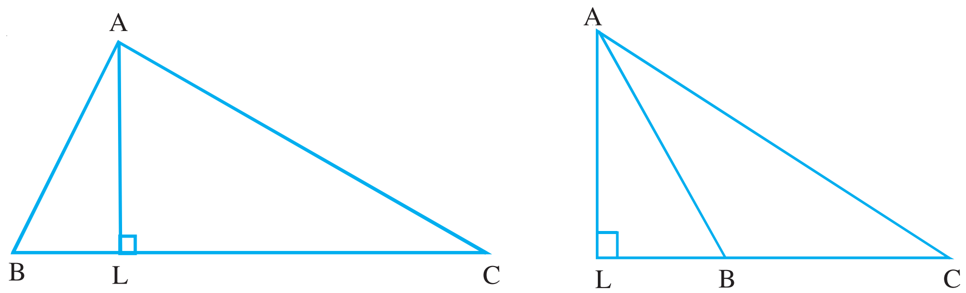


Fig. 12.14

Perpendicular drawn from a vertex of a triangle to the oposite side is called its altitude. How many altitudes can be drawn in a triangle? There are three vertices in a triangle, so we can draw three of its altitudes. AL is one of these altitudes. Now we draw the second altitude BM, which intersects the first altitude at a point H (see Fig. 12.15). We also draw the third altitude CN and observe that it also passes through the point H (Fig. 12.16). This shows that the three altitudes of the triangle pass through the same point.

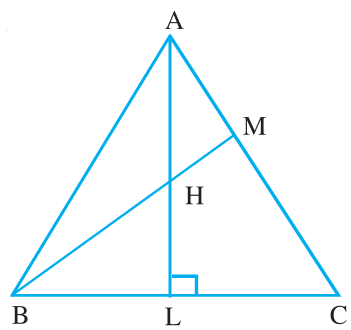


Fig. 12.15

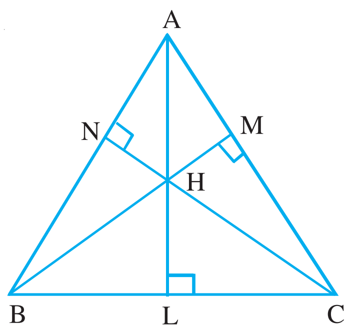


Fig. 12.16

We may take any type of triangle and draw its three altitudes. We always find that the three altitudes of a triangle are concurrent.

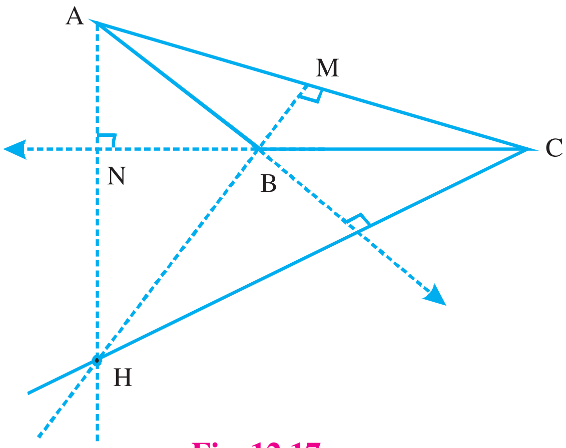


Fig. 12.17

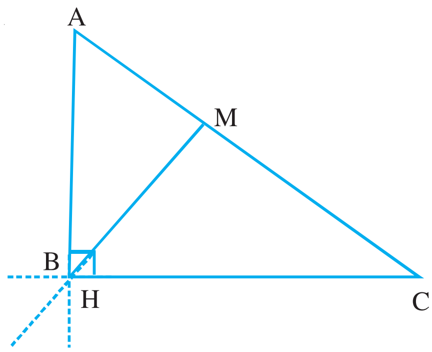


Fig. 12.18

Thus we conclude that:

In a triangle, the three altitudes pass through the same point, that is, they are concurrent.

The point of concurrency is called the 'Orthocentre' of the triangle.

Again observe that the orthocentre will be

1. in the interior of the triangle for an acute triangle (Fig. 12.16)
2. in the exterior of the triangle for an obtuse triangle (Fig. 12.17)
3. at the vertex containing the right angle for a right triangle (Fig. 12.18)

12.1.4 Medians of a Triangle

In ΔABC, AD joins the vertex A to the mid point D of the opposite side BC (Fig. 12.19)

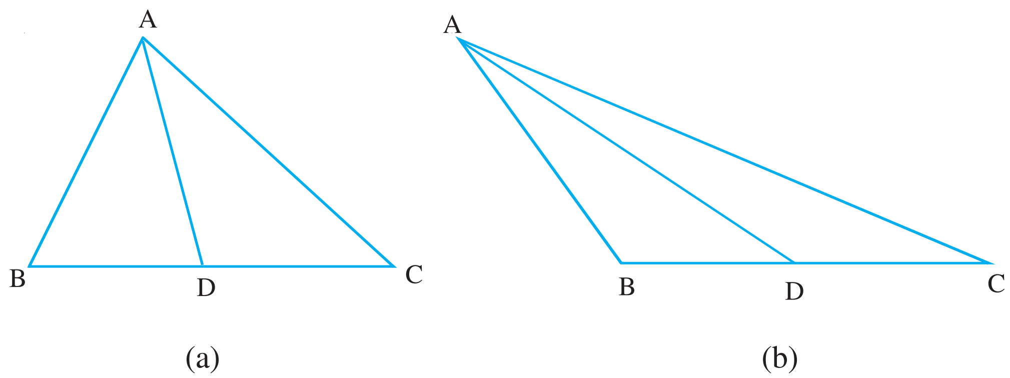
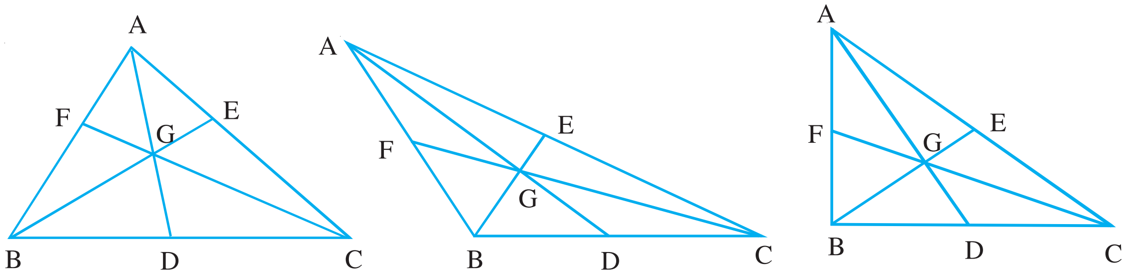


Fig. 12.19

A line joining a vertex to the mid point of the opposite side of a triangle is called its median. Clearly, three medians can be drawn in a triangle. AD is one of the medians. If we draw all the three medians in any triangle, we always find that the three medians pass through the same point [Fig. 12.20 (a), (b), (c)]



(a) (b) (c)

Fig. 12.20

Here in each of the triangles ABC given above (Fig. 12.20) the three medians AD, BE and CF are concurrent at G.. In each triangle we measure the parts into which G divides each median. On measurement, we observe that

$$AG = 2GD, \; BG = 2GE$$

and $$CG = 2\,GF$$

that is, the point of concurrency G divides each of the medians in the ratio 2 : 1.

Thus we conclude that:

Medians of a triangle pass through the same point, which divides each of the medians in the ratio 2 : 1.

The point of concurrency G is called the 'centroid' of the triangle.

ACTIVITY FOR YOU

Cut out a triangle from a piece of cardboard. Draw its three medians and mark the centroid G of the triangle. Try to balance the triangle by placing the tip of a pointed stick or a needle of compasses below the point G or at G. If the position of G is correctly marked then the weight of the triangle will balance at G (Fig. 12.21).

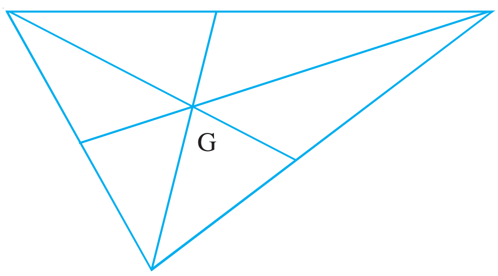


Fig. 12.21

Can you reason out, why the point of concurrency of the medians of a triangle is called its centroid. It is the point where the weight of the triangle is centered or it is the point through which the weight of the triangle acts.

We consider some examples using these concepts.

Example 12.1: In an isosceles triangle, show that the bisector of the angle formed by the equal sides is also a perpendicular bisector, an altitude and a median of the triangle.

Solution: In ΔABD and ΔACD,

$AB = AC$ (Given)

$\angle BAD = \angle CAD$ [$\because$ AD is bisector of $\angle A$]

and $AD = AD$

$\therefore$ $\Delta ABD \cong \Delta ACD$

$\therefore$ $BD = CD$

$\Rightarrow$ AD is also a median

$\Rightarrow$ Also $\angle ADB = \angle ADC = 90^\circ$

$\Rightarrow$ AD is an altitude

Since, $BD = DC$,

AD is perpendicular bisector of side BC.

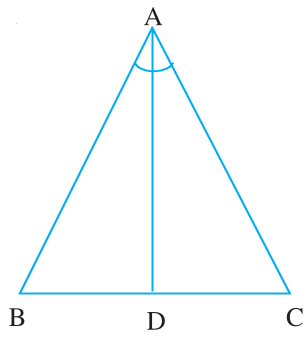


Fig. 12.22

Example 12.2: In an equilateral triangle, show that the three angle bisectors are also the three perpendicular bisectors of sides, three altitudes and the three medians of the triangle.

Solution: Since $AB = AC$

Therefore, AD, the bisector of $\angle A$ is also a perpendicular bisector of BC, an altitude and a median of the ΔABD.

(Refer Example 12.1 above)

Similarly, since $AB = BC$ and $BC = AC$

$\therefore$ BE and CF, angle bisectors of $\angle B$ and $\angle C$ respectively, are also perpendicular bisectors, altitudes and medians of the ΔABC.

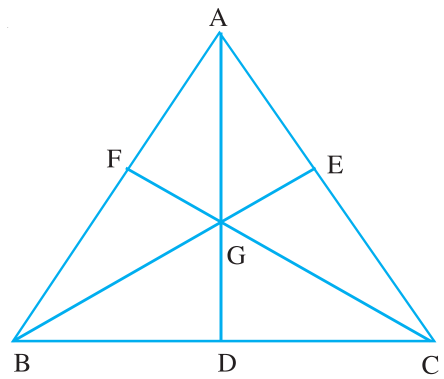


Fig. 12.23

Example 12.3: Find the circumradius of circumcircle and inradius of incircle of an equilateral triangle of side *a*.

Solution: We draw perpendicular from the vertex A to the side BC.

AD is also the angle bisector of $\angle A$, perpendicular bisector of side BC and a median joining vertex to the midpoint of BC.

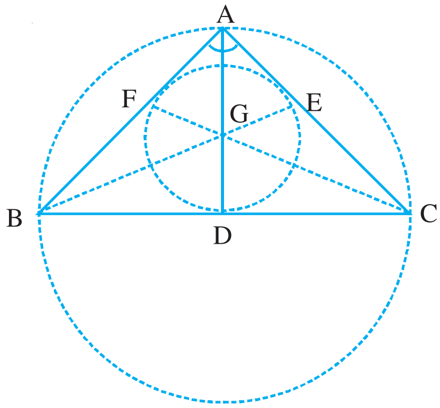


Fig. 12.24

$\therefore \quad AD = \frac{\sqrt{3}}{2}a,$ as BC = a

$\Rightarrow \quad$ AG = circumradius in this case $= \frac{2}{3} \times \frac{\sqrt{3}}{2} a = \frac{\sqrt{3}}{3} a$

and $\quad$ GD = inradius in this case $= \frac{1}{3} \times \frac{\sqrt{3}}{2} a = \frac{\sqrt{3}}{6} a$.

CHECK YOUR PROGRESS 12.1

1. In the given figure BF = FC, ∠BAE = ∠CAE and ∠ADE = ∠GFC = 90°, then name a median, an angle bisector, an altitude and a perpendicular bisector of the triangle.

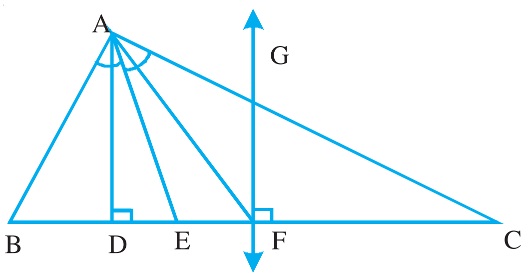


Fig. 12.25

2. In an equilateral triangle show that the incentre, the circumcentre, the orthocentre and the centroid are the same point.

3. In an equilateral ΔABC (Fig. 12.26). G is the centroid of the triangle. If AG is 4.8 cm, find AD and BE.

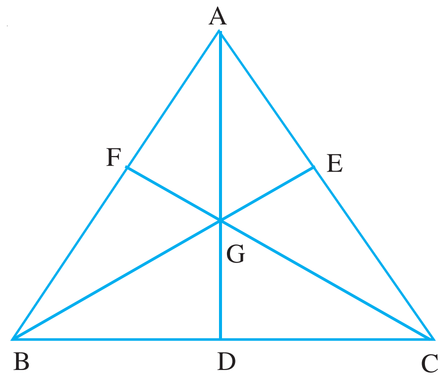


Fig. 12.26

4. If H is the orthocentre of ΔABC, then show that A is the orthocentre of the ΔHBC.

5. Choose the correct answers out of the given alternatives in the following questions:

   (i) In a plane, the point equidistant from vertices of a triangle is called its

   (a) centroid (b) incentre

   (c) circumcentre (d) orthocentre

   (ii) In the plane of a triangle, the point equidistant from the sides of the triangle is called its

   (a) centroid (b) incentre

   (c) circumcentre (d) orthocentre

LET US SUM UP

- Three or more lines in a plane which intersect each other in exactly one point are called concurrent lines.
- A line which bisects an angle of a triangle is called an angle bisector of the triangle.
- A line which bisects a side of a triangle at right angle is called the perpendicular bisector of the side of the triangle.
- A line drawn perpendicular from a vertex of a triangle to its opposite side is called an altitude of the triangle.
- A line which joins a vertex of a triangle to the mid-point of the opposite side is called a median of the triangle.
- In a triangle

  (i) angle bisectors are concurrent and the point of concurrency is called **incentre.**

(ii) perpendicular bisectors of the sides are concurrent and the point of concurrency is called **circumcentre.**

(iii) altitudes are concurrent and the point of concurrency is called **orthocentre.**

(iv) medians are concurrent and the point of concurrency is called **centroid,** which divides each of the medians in the ratio 2 : 1.

TERMINAL EXERCISE

1. In the given Fig. 12.27, D, E and F are the mid points of the sides of ΔABC. Show that $BE + CF > \frac{3}{2}BC$.

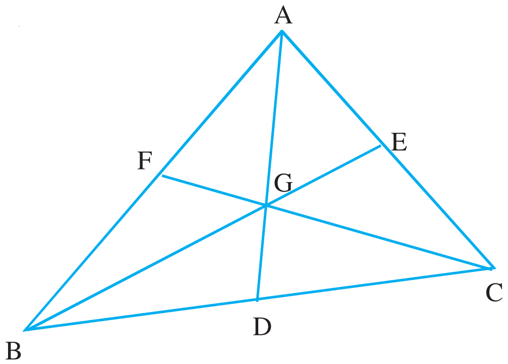


Fig. 12.27

2. ABC is an isoceles triangle such that AB = AC and D is the midpoint of BC. Show that the centroid, the incentre, the circumcentre and the orthocentre, all lie on AD.

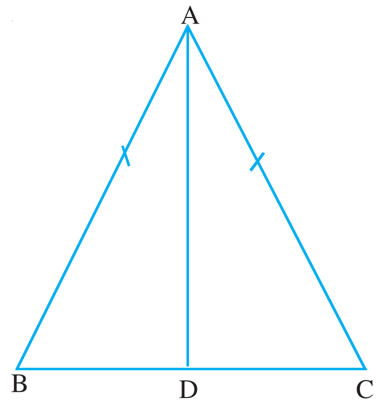


Fig. 12.28

3. ABC is an isoceles triangle such that AB = AC = 17 cm and base BC = 16 cm. If G is the centroid of ΔABC, find AG.

4. ABC is an equilateral triangle of side 12 cm. If G be its centroid, find AG.

ACTIVITIES FOR YOU

1. Draw a triangle ABC and find its circumcentre. Also draw the circumcircle of the triangle.
2. Draw an equilateral triangle. Find its incentre and circumcentre. Draw its incircle and circumcircle.
3. Draw the circumcircle and the incircle for an equilateral triangle of side 5 cm.

ANSWERS TO CHECK YOUR PROGRESS

12.1

1. Median - AF, Angle bisector AE

   Altitude - AD and perpendicular bisector - GF

3. AD = 7.2 cm, BE = 7.2 cm

5. (i) (c)   (ii) (b)

ANSWERS TO TERMINAL EXERCISE

3. AG = 10 cm

4. AG = $4\sqrt{3}$ cm